Non-approximability of the fall achromatic number optimization problem

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Abstract. We focus on fall colorings of graphs, i.e., proper colorings in which each vertex of any color sees each other color in its neighborhood. Such fall colorings are particular cases of b-colorings. If a fall coloring exists in a n vertices graph G, we denote respectively the minimum and maximum cardinality of a fall coloring in G by $\chi_f(G)$ and $\psi_f(G)$. We mainly show that the problem of determining $\psi_f(G)$ is NP-complete and can not be approximated within $\frac{n^{1-\epsilon}}{4}$ for any $\epsilon > 1$ unless $P = NP$. This result is related to the existence of graphs in which for some k, $\chi_f(G) < k < \psi_f(G)$ there is no fall coloring with cardinality k (such a graph is not *f-continuous*). Moreover, answering two open questions of Dunbar et al., we show that for any integer set S, there exists a graph G which set of cardinalities of fall colorings is S and that the problem of knowing if a given graph is f-continuous is NP-complete.

Résumé. Nous nous intéressons aux fall colorations de graphes. Une fall coloration est une coloration propre où chaque sommet d'une couleur quelconque voit toutes les autres couleur dans son voisinage. Une telle coloration est un cas particulier de la b-coloration. Si un graphe G admet au moins une fall coloration, nous notons la cardinalité respectivement minimum et maximum d'une fall coloration de G par $\chi_f(G)$ et $\psi_f(G)$. Nous prouvons principalement que le problème de déterminer $\psi_f(G)$ est NP-complet et ne peut pas être approché avec un facteur $\frac{n^{1-\epsilon}}{4}$ pour tout $\epsilon > 1$ à moins que $P = NP$. Ce résultat est lié à l'existence de graphes qui n'admettent pas certaines fall coloration de cardinalité k, $\chi_f(G) < k < \psi_f(G)$ (de tel graphes sont dits non *f-continus*). De plus, nous prouvons que pour n'importe quel ensemble S de nombres entiers, il existe un graphe G qui posséde une fall coloration avec k couleurs si et seulement si $k \in S$. Nous montrons enfin que le problème de savoir si un graphe est f-continu est un problème NP-complet.

1 Introduction

We investigate relations between some interpolation properties and various complexity and non-approximability aspects of some graph colorings. Consider a (k)-coloring (i.e., a proper coloring with cardinality k) in a graph G [2]. A way to obtain from it a $(k-1)$-coloring consists in changing the color of each vertex of a same chosen color. Such changing is not possible if for any used color in the (k)-coloring there is at least one vertex (called *colorful* vertex) with this color having each other color in its neighborhood. A (k)-coloring with this property is called a (k)-b-coloring of G [6]. It is clear that any $(\chi(G))$-coloring of G is a b-coloring. Thus, mainly studies focus on the maximal cardinality $b(G)$ of a b-coloring of G. A particular case of b-colorings called *fall colorings* have also been studied [7]. Such colorings are b-colorings in which each vertex is colorful. We call fall chromatic number and fall achromatic number and we denote respectively the minimum and maximum cardinalities of a fall coloring of G by $\chi_f(G)$ and $\psi_f(G)$. Note that fall colorings do not exist in any graph and Dunbar et al. have shown that the problem of knowing if a given graph admits a fall coloring is NP-complete [7].

An interesting property of b-colorings and fall colorings concerns interpolation [4]. Indeed, there exist graphs G such that for some integers k, $\chi(G) < k < b(G)$, there is no (k)-b-coloring. We say that such a graph G is not *b-continuous*. In fact, we have shown in [3] that for any integer set, there exists a graph which set of b-colorings cardinalities (called b-spectrum) is this given set. We show that the same property holds for fall colorings and we show that the problem of knowing if a given graph is f-continuous is NP-complete, as we did in [3] about b-continuity.

Given an integer k, determining whether $b(G)$ is greater or equal to k is a NP-complete problem [6, 8]. Answering a question of Dunbar et al. [7], we show in this paper that so it is for $\psi_f(G)$. Note that we also answer another question of [7] by showing that for any integer n there exists a graph such that $\chi_f(G) - \chi(G) \geq n$. Thus, considering these results about complexity, the question we want to deal with is to know how the existence of some not f-continuous graphs influences approximability behaviors of the problem of determining $\psi_f(G)$. To our knowledge, the only first result about non-approximability concerns b-coloring and does not use the b-continuity property [9].

In this paper, we focus on the link between the fact that some graph (with at least one f-coloring) are not f-continuous and that the problem of determining $\psi_f(G)$ can not be approximated by a constant ratio. In fact, we define a class of graph with n vertices with f-spectrum $\{2\}$ or $\{2, \alpha\}$, with $\alpha \in \theta(n)$, in which knowing if there exists a fall α-coloring is a NP-complete problem. This shows that there is no approximation algorithm with ratio less than $\frac{n^{1-\epsilon}}{4}$.

The paper is organized as follows. In the next section, we show that for any finite integer set S, there exists a graph with f-spectrum S(i.e., S is the set of all fall coloring cardinalities of the graph). Then, in Section 3, we present our complexity results. Finally, answering an open question of Dundar et al [7].

2 A graph with a given f-spectrum

To prove Theorem 2, we first need to define a particular graph product. For arbitrarily graphs G and H, we define the *categorical product* of G and H to be the graph $G \times H$ with vertices $\{(u, v) : u \in G, v \in H\}$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times H$ if and only if u_1 is adjacent to u_2 in G and v_1 is adjacent to v_2 in H. Figure 1 gives the categorical product $K_2 \times K_4$. In [7] Dunbar et al. have shown that:

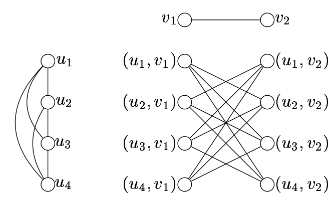

Fig. 1. The categorical product $K_2 \times K_4$

Theorem 1. *[7] The f-spectrum of the graph $K_{n_1} \times K_{n_2}$ where $n_1 \geq 2$ and $n_2 \geq 2$ is the set $\{n_1, n_2\}$.*

This theorem does not generalize to categorical products of three or more complete graphs. Therefore, in an arbitrarily specified set $S = \{n_0, n_1, \ldots, n_p\}$ of positive integers, the categorical product G of complete graphs K_{n_i} for every $n_i \in S$, has fall n_i-colorings, for every $n_i \in S$, but graph G has also other fall k-colorings for some $k \notin S$.

Theorem 2. *For any finite nonempty set $S \subset (\mathbb{N} \setminus \{0, 1\})$, there exists a graph G such that $S_f(G) = S$.*

The remaining of this section is devoted to the proof of Theorem 2. We deal first with the particular cases where $|S| \leq 2$. If $S = \{n\}$, then the complete graph K_n has the required property. If $S = \{2, n\}$, then by Theorem 1, S is the f-spectrum of the graph $K_2 \times K_n$. To prove the case $|S| > 2$, we consider two steps.

Step 1: Set S with $\arg\min S = 2$

Let $S = \{n_0, n_1, n_2, \ldots, n_p\}$ such that $n_0 = 2 < n_1 < n_2 < \ldots < n_p$ and $2 \leq p$. We initialize G_S with graph $K_2 \times K_{n_1} \times \ldots \times K_{n_p}$. Moreover we add a new vertex v to G_S and connect v to vertex in B defined by : $B = \{(1, i_1, \ldots, i_1) : 1 \leq i_1 \leq n_1 - 1\} \cup_{\ell=2}^{p-1} \{(n_0 - 1, n_1 - 1, n_2 - 1, \ldots, n_{\ell-1} - 1, \beta, \ldots, \beta) : n_{\ell-1} \leq \beta \leq n_\ell - 1\}$. In Figure 2, a partial construction of the graph G_S for $S = \{2, 4, 6\}$ is given, v is adjacent to vertices in $\{(1, 1, 1), (1, 2, 2), (1, 3, 3), (1, 3, 4), (1, 3, 5)\}$.

By definition of the categorical product, we have

Lemma 1. *Two vertices $x = (x_0, x_1, \ldots, x_p)$ and $y = (y_0, y_1, \ldots, y_p)$ of graph G_S are adjacent if and only if $x_i \neq y_i$ for every i, $0 \leq i \leq p$.*

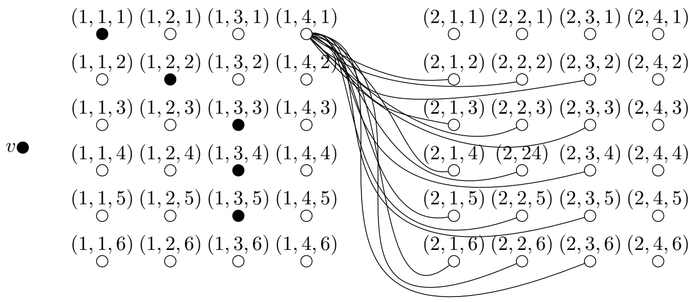

Fig. 2. Partial construction of the G for $S = \{2, 4, 6\}$. Only the edges between vertex $(1, 4, 1)$ and its neighborhood are drawn.

Let us prove that G_S satisfies the following lemma:

Lemma 2. *For every $0 \leq i \leq p$, G_S has a fall n_i-coloring.*

Proof. G_S is a bipartite graph because all edges of G_S have exactly one extremity in vertex set $\{(1, x_1, x_2, \ldots, x_p) : 1 \leq x_i \leq n_i, 1 \leq i \leq p\}$. So G_S has a fall 2-coloring.

Given an integer i such that $1 \leq i \leq p$, a fall n_i-coloring π of G_S is constructed as follows. The color of vertex v is the color n_i. For $x_i = 1, \ldots, n_i$, vertices in $\{(x_0, x_1, \ldots, x_i, \ldots, x_p) : 1 \leq x_j \leq n_j, 0 \leq j \leq p \wedge j \neq i\}$ and $1 \leq x_j \leq n_j$ are colored with the color x_i. We check first that coloring π is proper. If two vertices $x = (x_0, x_1, \ldots, x_p)$ and $y = (y_0, y_1, \ldots, y_p)$ of graph G_S are colored with the same color then $x_i = y_i$. So, x and y are not adjacent (see Lemma 1). Moreover, vertex v is not adjacent to $x = (x_0, x_1, \ldots, x_p)$ if $x_i = n_i$. So π is proper. Furthermore, the vertices in $V(G_S) \setminus \{v\}$ are colorful because each vertex $x = (x_0, x_1, \ldots, x_\ell, \ldots, x_p)$ of color x_ℓ iscolo adjacent to vertex in $x' = (x_0, x_1, \ldots, x_j, \ldots, x_p)$ of color x_j with $j \neq \ell$. Since v is adjacent to the vertices $(1, j_1, j_1, \ldots, j_1), (1, n_1 - 1, j_2, \ldots, j_2) \ldots (1, n_1 - 1, \ldots n_{i-1} - 1, j_i, \ldots, j_i)$ where $1 \leq j_1 \leq n_1 - 1$ and $n_{h-1} - 1 \leq j_h \leq n_h - 1$ with $2 \leq h \leq i$, vertex v is colorful. So coloring π is a fall a n_i-coloring of G. This concludes the proof of this lemma. □

It remains to prove that if G_S has a fall k-coloring, then $k \in S$. For a given $x_0, x_1, \ldots x_i$, where $1 \leq x_j \leq n_j$ and $0 \leq j \leq i \leq p-1$, let $A_{i+1}(x_0, x_1, \ldots, x_i)$ be a subset of $V(G_S)$ such that $A_{i+1}(x_0, x_1, \ldots, x_i) = \{(x_0, x_1, \ldots, x_i, x_{i+1}, \ldots, x_p) : 1 \leq x_r \leq n_r \wedge i + 1 \leq r \leq p\}$.

We show now the following lemma.

Lemma 3. *For every fall k-coloring of graph G_S such that $k < n_{r+1}$, all the vertices in $A_{r+1}(x_0, x_1, \ldots, x_r)$ are colored with the same color.*

Proof. Let $\Gamma(A)$ be the neighborhood of a vertex set A in G_S. First, we consider the case where $r = p - 1$. Assume that G has a fall k-coloring with $k < n_p$. We consider two sub-cases: $x_0 = 2$ and $x_0 = 1$.

Case 1: $x_0 = 2$. Vertex v and $A_p(2, x_1, \dots, x_{p-1})$ are in the same class of the bipartition. As $|A_p(2, x_1, \dots, x_{p-1})| = n_p > k$, at least two vertices u_1 and u_2 of $A_p(2, x_1, \dots, x_{p-1})$ are colored with the same color denoted by c. Moreover, by construction of G_S, we have $\Gamma(u_1) \cup \Gamma(u_2) = \Gamma(A_{i+1}(x_0, x_1, \dots, x_i))$. Hence the vertices of $\Gamma(A_{i+1}(x_0, x_1, \dots, x_i))$ can not be colored by the color c. So, to be colorful, all the vertices of $A_p(x_0, x_1, \dots, x_i)$ must be colored c.

Case 2: $x_0 = 1$. The vertex $x = (1, x_1, \dots, x_{p-1}, n_p)$ in $A_p(1, x_1, \dots, x_{p-1})$ is not adjacent to v and its color is c. As vertex x must be colorful, then for every color $c' \neq c$, x is adjacent to a vertex $y = (2, y_1, \dots, y_p)$ colored c'. By Case 1, if $y = (2, y_1, \dots, y_p)$ is colored c', then all the vertices in $A_p(2, y_1, \dots, y_{p-1})$ are colored c'. This implies that every vertex of the set $A_p(1, x_1, \dots, x_{p-1})$ are colored with the color c' on its neighborhood for every color $c' \neq c$. This means that all the vertices of $A_p(1, x_1, \dots, x_{p-1})$ are colored with color c.

The case where $r = p - 1$ holds. Assume that the lemma holds for $r = p-1, \dots, i+1$, and let us prove it for $r = i$. Assume that G has a fall k-coloring with $k < n_{i+1}$. We have

$$A_{i+1}(x_0, x_1, \dots, x_i) = \bigcup_{1 \leq x_i \leq n_{i+1}} A_{i+2}(x_0, x_1, \dots, x_{i+1}.)$$

Since $k < n_{i+1} < n_{i+2}$, by hypothesis, all the vertices in $A_{i+2}(x_0, x_1, \dots, x_{i+1})$ are colored with the same color. We consider two sub-cases: $x_0 = 2$ and $x_0 = 1$.

Case a: $x_0 = 2$. Since $k < n_{i+1}$, all the vertices of at least two subsets $A_{i+2}(2, x_1, \dots, x^1_{i+1})$ and $A_{i+2}(2, x_1, \dots, x^2_{i+1})$ of $A_{i+1}(2, x_1, \dots, x_i)$ must be colored with the same color denoted color c. Moreover, it is easy to see that the neighborhood of any two subsets $A_{i+2}(2, x_1, \dots, x^1_{i+1})$ and $A_{i+2}(2, x_1, \dots, x^2_{i+1})$ is equal to the neighborhood of $A_{i+1}(2, x_1, \dots, x_i)$. This implies that the color c will not appears in the neighborhood of the set $A_{i+1}(2, x_1, \dots, x_i)$. So, to be colorful, the vertices of $A_{i+1}(2, x_1, \dots, x_i)$ would have to be colored c.

Case b: $x_0 = 1$. The vertex $x = (1, x_1, \dots, x_{p-1}, n_p)$ of $A_{i+1}(1, x_1, \dots, x_i)$ is not adjacent to v and its color is c. Vertex x is colorful. So for every color $c' \neq c$, x is adjacent to at least one vertex $y = (2, y_1, \dots y_i, \dots, y_p)$ colored c'. By Case a, all the vertices of $A_{i+1}(2, y_1, \dots, y_i)$. But every vertex in $A_{i+1}(x_0, x_1, \dots, x_i)$ has at least one neighbor in $A_{i+1}(2, y_1, \dots, y_i)$, Hence the color c' can not be assigned to any vertex of the set $A_{i+1}(1, x_1, \dots, x_i)$, and this for every $c' \neq c$. This implies that all the vertices in $A_{i+1}(x_0, x_1, \dots, x_i)$ are colored with color c. This concludes this lemma.

□

Now, we can determine the f-spectrum of graph G_S:

Lemma 4. $S_f(G_S) = S$

Proof. By Property 2, we know that $S \subseteq S_f(G_S)$ Now, we will prove that, if G_S has a fall k-coloring with $n_i \leq k < n_{i+1}$ and $1 \leq i \leq p-1$, then $k = n_i$. Assume that such a fall k-coloring exists. Vertex v has some neighbors in exactly $n_{i-1} - 1$ distinct sets $A_{i+1}(x_0, x_1, \dots, x_i)$. Indeed, vertex v has some neighbors in the sets

$A_{i+1}(1, j_1, \ldots, j_1), \ldots, A_{i+1}(1, n_1 - 1, \ldots, n_{h-1} - 1, j_h, \ldots j_h), \ldots, A_{i+1}(1, n_1 - 1, \ldots, n_{i-1} - 1, j_i)$, with $n_{h-1} - 1 \leq j_h \leq n_h - 1$ and $1 \leq h \leq i$. Moreover, by Lemma 3, for every $x_1, x_2, \ldots, x_i$ with $x_0 = 1, 2$, $1 \leq x_j \leq n_j$ and $1 \leq j \leq i$, all the vertices in $A_{i+1}(x_0, x_1, \ldots, x_i)$ are colored with the same color. This means that vertex v has at most $n_i - 1$ colors on its neighborhood. So, v is colorful if and only if $k = n_i$. □

So, graph G_S are build such that $S_f(G_S) = S$ and $\arg\min S = 2$. Now we generalize this result.

Step 2: Set $S = \{n_i : 2 \leq n_i \wedge 0 \leq i \leq p\}$
First we give a simple result about combining the f-spectrums of two graphs. The *join* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets is the graph $G = (V, E)$ defined by $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup R$, where $R = \{(v_1, v_2) : v_1 \in V_1, v_2 \in V_2\}$.

Lemma 5. *Let G_1 and G_2 be two graphs. The f-spectrum of the join of G_1 and G_2 is $\{i + j : i \in S_f(G_1), j \in S_f(G_2)\}$*

Proof. The edges added between the vertex sets of G_1 and G_2 prohibit colors from appearing in both sets. Thus the proper colorings consist of proper colorings of G_1 and of G_2 using disjoint sets of colors. □

Let $S = \{n_0, n_1, n_2, \ldots, n_p\}$ such that $n_{i-1} < n_i$ for all $i \in [1, \ldots, p]$. Using the previous case, there exists a graph G such that $S_f(G) = \{2, n_1 - n_0 + 2, n_2 - n_0 + 2, \ldots, n_p - n_0 + 2\}$. By Lemma 5, the f-spectrum of the graph join of G and K_{n_0-2} is S. And the proof of Theorem 2 is ended.

3 Complexity

This section is devoted to study the complexity of the problems connected to the fall coloring. First, we study the complexity to the following problem:
FALL K-COLORING (FKC)
Instance: Graph G having a fall coloring and an integer K.
Question: Does G have a fall α-coloring such that $\alpha \geq K$?

Next, we consider its optimization versions defined as follows:
MAXIMUM FALL K-COLORING (MFKC)
Instance: Graph G having a fall coloring.
Solution: a α-coloring of G
Measure: α

Let $OPT(x)$ denote the optimal value for any arbitrary instance x of *MFKC* and let $B(I)$ the solution found by an algorithm B. Let a function $\alpha : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$. We say that a polynomial-time algorithm B is an α-approximate algorithm for *MFKC* iff for every instance x of *MFKC* of size n, B produces a solution in the range $[OPT(x)/\alpha(n), OPT(x)]$. We say that *MFKC* is approximable within a factor α if such an algorithm exists. The remaining of this section proves that $r-$approximability with $r < \left(\frac{n^{1-\epsilon}}{4}\right)$, becomes computationally intractable.

First, we present the polynomial time construction from an instance of Problem *NOT-ALL-EQUAL 3-SATISFIABILITY* to a graph G_t (the idea of the construction is based on the work described in [7]). Next, we define the f-spectrum of graph G_t. Finally, we give some complexity results about the fall coloring.

3.1 Polynomial Transformation

The NP-complete problem *NOT-ALL-EQUAL 3-SATISFIABILITY (NAE-3-SAT)* is defined as follow [5, 10]:
NOT-ALL-EQUAL 3-SATISFIABILITY (NAE-3-SAT)
Instance: Set U of variables, collection C of disjunctive clauses over U such that each clause $C_i \in C$ has $|C_i| = 3$.
Question: Is there a truth assignment for U such that each clause in C has at least one true literal and at least one false literal?

We give now a polynomial time transformation called $\mathcal{A}$ takes as input an instance $I = \langle U\{u_1, \ldots, u_k\}, C = \{C_1, \ldots, C_p\}\rangle$ of *NAE-3-SAT* and an integer t. It constructs a graph G_t, a part of instance of *FKC*. Figure 3 gives an example the resulting graph from algorithm $\mathcal{A}$.

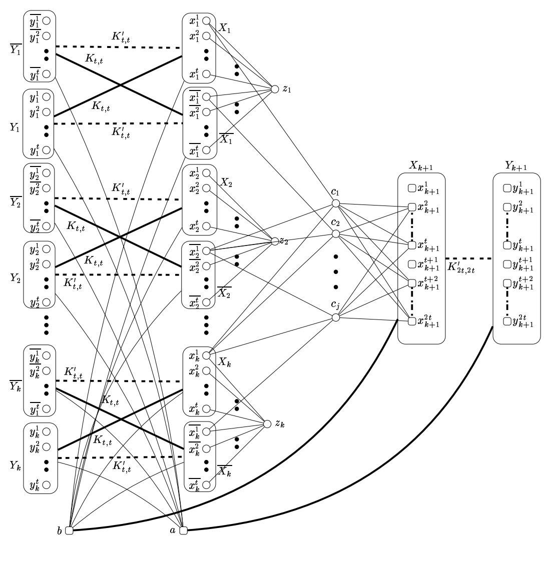

Fig. 3. An example of graph G_t such that instance I has $C_1 = (u_1, \overline{u_2}, u_k)$

Algorithm $\mathcal{A}$.

Input: Instance I of NAE-3-SAT $\langle U = \{u_1, \ldots, u_k\}, C = \{C_1, \ldots, C_p\}\rangle$
an integer t

Output: Graph $G_t = (V_t, E_t)$

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1. $X_{k+1} \leftarrow \{x_{k+1}^1, x_{k+1}^2, \ldots, x_{k+1}^{2t}\}$, $Y_{k+1} \leftarrow \{y_{k+1}^1, y_{k+1}^2, \ldots, y_{k+1}^{2t}\}$,
2. $V_t \leftarrow \{a, b\} \cup X_{k+1} \cup Y_{k+1}$
3. $E_t \leftarrow \{(y_{k+1}^{\ell}, x_{k+1}^{\ell'}), (y_{k+1}^{\ell}, a), (x_{k+1}^{\ell}, b) : 1 \leq \ell \leq 2t \wedge 1 \leq \ell' \leq 2t \wedge \ell \neq \ell'\}$
4. for each variable $u_i \in U$ do
   (a) $X_i \leftarrow \{x_i^1, x_i^2, \ldots, x_i^t\}$, $\overline{X_i} \leftarrow \{\overline{x_i^1}, \overline{x_i^2}, \ldots, \overline{x_i^t}\}$
   (b) $Y_i \leftarrow \{y_i^1, y_i^2, \ldots, y_i^t\}$, $\overline{Y_i} \leftarrow \{\overline{y_i^1}, \overline{y_i^2}, \ldots, \overline{y_i^t}\}$,
   (c) $V_t \leftarrow V_t \cup Y_i \cup \overline{Y_i} \cup X_i \cup \overline{X_i} \cup \{z_i\}$
   (d) $E_t \leftarrow E_t \cup \{(\overline{y_i^{\ell}}, \overline{x_i^{\ell'}}), (y_i^{\ell}, x_i^{\ell'}) : 1 \leq \ell \leq t \wedge 1 \leq \ell' \leq t\}$
   (e) $E_t \leftarrow E_t \cup \{(\overline{y_i^{\ell}}, x_i^{\ell'}), (y_i^{\ell}, \overline{x_i^{\ell'}}) : 1 \leq \ell \leq t \wedge 1 \leq \ell' \leq t \wedge \ell \neq \ell'\}$
   (f) $E_t \leftarrow E_t \cup \{(\overline{x_i^{\ell}}, z_i), (x_i^{\ell}, z_i), (\overline{x_i^{\ell}}, b), (x_i^{\ell}, b), (\overline{y_i^{\ell}}, a), (y_i^{\ell}, a) : 1 \leq \ell \leq t\}$
5. For each clause $C_d \in C$ do,
   (a) $V_t \leftarrow V_t \cup \{c_d\}$
   (b) $E_t \leftarrow E_t \cup \{(c_d, x_{k+1}^{\ell}) : 2 \leq \ell \leq 2t \wedge \ell \neq t+1\}$
   (c) for each literal u_α in C_d do $E_t \leftarrow E_t \cup \{(c_d, x_\alpha^1)\}$
6. return G_t

Since the total number of vertices of G_t is $4t(k+1) + k + p + 2$, algorithm $\mathcal{A}$ runs in polynomial time in term of t and of the size of instance I. The next section is devoted to study the f-spectrum of G_t.

3.2 The f-spectrum of G_t

Now, we will give the f-spectrum of G_t. We will use is Section 3.3. This f-spectrum depends on the instance I of *NAE-3-SAT* and on an integer t.

Proposition 1. *Let* $I = \langle U, C\rangle$ *be an instance of* NAE-3-SAT. *Let* G_t *be the result of algorithm* $\mathcal{A}$ *having* I *and an arbitrary integer* t *as input.*

1. *The f-spectrum of* G_t *is* $\{2, 2t+1\}$ *if set* U *of variables has a truth assignment with desired properties*
2. *The f-spectrum of* G_t *is* $\{2\}$ *otherwise*

Proof. First, it is easy to see that graph G_t is a bipartite graph because all edges have only one vertex in $\cup_{i=1}^{k} \left(X_i \cup \overline{X_i}\right) \cup X_{k+1} \cup \{a\}$. So, we deduce that:

Claim 1. G_t *has a fall* 2*-coloring.*

Moreover, graph G_t satisfies the following claim:

Claim 2. *The graph G_t has not any fall n-coloring for $n = 3, \dots, 2t$.*

Proof. We show Claim 2 by contradiction. Assume that G_t has a fall n-coloring with $2 < n < 2t+1$.

Without loss of generality, we assume that the color of vertex z_1 is n. The neighborhood of z_1 is $X_1 \cup \overline{X_1}$. Since z_1 is colorful, for each color c in $\{1, \dots, n-1\}$ there exists at least one vertex in $X_1 \cup \overline{X_1}$ colored c. Moreover, vertex b must be colored n, because it is adjacent to all the vertices of $X_1 \cup \overline{X_1}$.

Since $n - 1 < 2t$ and since $|X_1 \cup \overline{X_1}| = 2t$, at least two vertices in $X_1 \cup \overline{X_1}$ are colored with the same color, denoted by color α. This implies that color α can not be assigned to any vertex of $Y_1 \cup \overline{Y_1}$, because each vertex of $Y_1 \cup \overline{Y_1}$ has $2t-1$ neighbors in $X_1 \cup \overline{X_1}$. So, every vertex of $Y_1 \cup \overline{Y_1}$ has at least one neighbor in $X_1 \cup \overline{X_1}$ colored α.

Since there are at least $n-1$ distinct colors in $X_1 \cup \overline{X_1}$, there exists at least one vertex u of $X_1 \cup \overline{X_1}$ colored α' with $\alpha' \neq \alpha$. But vertex u can not be colorful. Indeed, vertex u has not any neighbor of color α. This leads to contradiction with the definition of fall coloring. This conclude the proof of Claim 2.

From Claims 1 and 2, it remains to prove that set U of variables has a truth assignment with desired properties if and only if G_t has a fall $2t+1$-coloring.

Claim 3. *If set U of variables has a truth assignment with desired properties, then G_t has a fall $2t+1$-coloring.*

Proof. Assume first that I has a satisfying truth assignment $f : X \to \{T, F\}$. Color the vertices $\{a, b, z_i, c_\ell : 1 \leq i \leq k \wedge 1 \leq \ell \leq p\}$ with the color $2t+1$. Finally, for $i = 1, \dots, k$, if $f(X_i) = T$, color the vertices $\{x_i^\ell, y_i^\ell, \overline{x_i^\ell}, \overline{y_i^\ell}\}$ with the colors $\{\ell, \ell+t, \ell+t, \ell\}$ respectively for $\ell = 1, \dots, t$. Otherwise, color the vertices $\{x_i^\ell, y_i^\ell, \overline{x_i^\ell}, \overline{y_i^\ell}\}$ with the colors $\{\ell+t, \ell, \ell, \ell+t\}$ respectively for $\ell = 1, \dots, t$. Moreover, color the vertices x_{k+1}^ℓ and y_{k+1}^ℓ with the color ℓ for $\ell = 1, \dots, 2t$. Clearly this coloring is a proper coloring. Moreover, by construction of the coloring, it is easy to see that every vertex not in $\{c_i : 1 \leq i \leq p\}$ is colorful. Now, it remains to check that all the vertices in $\{c_i : 1 \leq i \leq p\}$ are colorful. Let C_j be a clause in C. Vertex c_j is adjacent to vertices in $\{x_{k+1}^\ell, x_{k+1}^{\ell+t} : 1 < \ell \leq t\}$ and so it is adjacent to vertices of colors ℓ, $1 < \ell \leq t$ and to vertices of colors ℓ, $t+1 < \ell \leq 2t$ (by construction of the coloring). Since f is a satisfying truth assignment such that every clause C_j has at least one true literal and at least one false literal, vertex c_i is adjacent to vertices of colors $1, t+1$. So this coloring is a fall coloring. This concludes the proof of Claim 3.

Conversely, we prove the following claim:

Claim 4. *If G_t has a fall $2t+1$-coloring. Then, set U of variables has a truth assignment with desired properties.*

Proof. Assume that graph G_t has a fall $(2t+1)$-coloring. Without loss of generality, we assume that the color of vertex b is $2t+1$.

Let i be an integer between 1 and k. Since z_i is a colorful vertex and since z_i is only adjacent to vertices in $X_i \cup \overline{X_i}$, set $X_i \cup \overline{X_i}$ contains $2t$ distinct colors and it does not contain the color of z_i. Since vertex b is adjacent to $X_i \cup \overline{X_i}$, the color of b is also the color of z_i. So, for $i = 1, \dots, k$, the color of z_i is $2t + 1$. Since, the vertices of Y_{k+1} are of degree $2t$, the neighborhood of any vertex y of Y_{k+1} must have distinct colors. Otherwise y cannot be colorful. This implies that the vertices of X_{k+1} are colored by distinct colors. Indeed, if two vertices of X_{k+1} have the same color then there exists at least one vertex of Y_{k+1} with two neighbor of the same color. So the vertices of X_{k+1} have $2t$ distinct colors in X_{k+1}.

Let C_j be a clause in C. Since c_j is adjacent to vertices in $\bigcup_{i=1}^{k} X_i$, c_j is colored with the color $2t+1$. Moreover, for any clause C_j, vertex c_j is adjacent to $\{x_{k+1}^{\ell} : 1 < \ell \leq 2t \wedge \ell \neq t+1\}$. Since vertices in $\{x_{k+1}^{\ell} : 1 < \ell \leq 2t \wedge \ell \neq t+1\}$ have $2t - 2$ distinct colors, we assume without loss of generality that no vertex in this set of vertices is colored with the color 1 or the color $t + 1$. We define a function $f : U \rightarrow \{T, F\}$ by setting $f(u_i) = T$ if vertex x_i^1 is colored with color 1, otherwise $f(u_i) = F$. Since the coloring is a fall $(2t + 1)$-coloring, each vertex c_i is adjacent to at least one vertex of color 1 and at least one vertex of color $t + 1$. This function f defined here is a satisfying truth assignment with desired properties for *NAE-3-SAT*. Theses conclude the proof of Claim 4 and the proof of Proposition 1. □

3.3 Complexity results

We give now the main result of the paper concerning a non-approximation result about MFKC related to the fact that G_t is not f-continuous. Proposition 1 allows us to deduce that:

Theorem 3. *The problem* FKC *is NP-complete even if graph G is bipartite and $k = 3$. Its optimization problem* MFKC *is not approximable within $\frac{n^{1-\epsilon}}{4}$ for any $\epsilon > 0$ where n is the number of vertices, unless $P = NP$.*

Proof. First, it is easy to see that problem *FKC* belongs to NP. Moreover, let I be an instance of *NAE-3-SAT*. We get an instance I' of *FKC* by setting $k = 3$. Graph G of I is computed by algorithm $\mathcal{A}$ described in Section 3.1 with I and $t = 1$ as input. By Proposition 1, we can deduce that the problem *FKC* is NP-complete even if graph G is bipartite and $k = 3$.

Moreover, using the gap technique [1], by Proposition 1, we prove that no polynomial-time t-approximate algorithm for *MFKC* can exist unless $P = NP$. Now, we will compare t to n. In order to simplify this proof, we set $t > (k+p)^2$ and graph G_t is transformed to graph G' by adding vertices whose have the same neighborhood as vertex a such that $n = 4t(k + p)$ where n is the number of the new graph $(4t(k+p) \geq 4t(k+1)+k+p+2)$. This graph satisfies Proposition 1.

Arbitrarily choose an $\epsilon > 0$. Let the natural number $c > 2$ satisfy $1 - \frac{1}{c+1} \geq 1 - \epsilon$. Let $t = (k + p)^c$. So graph G' satisfies the following property:

1. G' is a graph with $4(k + p)^{c+1}$ vertices.

2. G' a fall $2t+1$-coloring if instance I is a positive instance.
3. G' a fall 2-coloring if instance I is a negative instance.

By computation, it is easy to see that $t = (\frac{n}{4})^{1-\frac{1}{c+1}}$ and $t \geq \frac{n^{1-\frac{1}{c+1}}}{4}$. Since $1 - \frac{1}{c+1} \geq 1 - \epsilon$ and since ϵ was arbitrarily chosen, problem *MFKC* is not approximable within $\frac{n^{1-\epsilon}}{4}$. □

Thus, the NP-completeness of FKC and the fact that MFKC does not belong to APX (unless P=NP) is related to the existence of not f-continuous graphs. Once could ask if similar results occur if we consider only f-continuous graphs. This is an open question, but we can give the following first result about the difficulty to determine if a graph is f-continuous. Let us call *f-CONTINUITY* the problem:

f-CONTINUITY
Instance: Graph G having a fall coloring.
Question: Does graph G f-continuous?

Theorem 4. *The problem* f-CONTINUITY *is NP-Complete.*

Proof. Problem *f-CONTINUITY* is in NP: Since for a graph G, for each integer k between 2 and n, a non-deterministic polynomial time algorithm can determine if there exists a fall (k)-coloring of G.

We prove that problem *f-CONTINUITY* is NP-hard. The reduction takes $I = \langle U\{u_1, \ldots, u_k\}, C = \{C_1, \ldots, C_j\}\rangle$ as input. First, G_1 is computed by algorithm $\mathcal{A}$ described in Section 3.1 taken I and $t = 1$ as input in polynomial time. Afterwards, graph G is constructed from the join of the graph G_1 and the graph $H = K_4 \times K_2$. It is easy that this transformation runs in polynomial time. To complete the proof, we show that this transformation is indeed a reduction: graph G is f-continuous if and only if U has a truth assignment with desired properties.

By Theorem 1, the f-spectrum of H is the set $\{2, 4\}$. Moreover, by Proposition 1, we know that the f-spectrum of G_1 is $\{2, 3\}$ if and only if U has a truth assignment with desired properties. From Lemma 5, we deduce that the f-spectrum of graph G is the set $\{4, 5, 6, 7\}$ if and only if U has a truth assignment with desired properties. And otherwise, the f-spectrum of G is the set $\{4, 6\}$. So G is f-continuous if and only if U has a truth assignment with desired properties. □

To end this paper, we answer an open question of Dundar et al [7].

Proposition 2. *For every n, there is a graph G such that $\chi_f(G) - \chi(G) > n$.*

Proof. We consider the graph G obtained from the complete bipartite graph $K_{n+3,n+3} = (U \cup V, E)$ by removing a perfect matching and adding an edge (u_1, u_2), between two vertices $u_1, u_2 \in U$. Clearly $\chi(G) = 3$, and it is easy to see that G has a fall $n+3$-coloring. To show that $\chi_f(G) = n+3$. It remains to prove that G do not have fall k-colorings for $3 < k < n+3$. Assume that G has such a fall k-coloring. As $k < |V|$, at most two vertices of V are colored with same color denoted by c. Since the neighborhood of any two vertices of V is the set U, color c can not be assigned to any vertex of U. This means that every vertex $v \in V$ must be colored with color c (if it is not the case, then v will not be colorful). This implies that no vertex $u \in U \setminus \{u_1, u_2\}$ is colorful, since all its neighbors are colored with the same color c. Therefore $\chi_f(G) = n+3$ and $\chi_f(G) - \chi(G) > n$. □

4 Conclusion

The main result of this paper given in Theorem 3 shows that problem MFKC can not approximable within $\frac{n^{1-\epsilon}}{4}$ for any $\epsilon > 0$. To our knowledge, this is the first result giving a relation between interpolation properties of a coloring (i.e., there exist not f-continuous graphs) and the non-approximability of its maximum colors cardinality. This result is directly deduced from the fact that there is a "hole" between this (possible) maximal number and before the last one. We also answer some open question from Hedetniemi et al concerning f-continuity. As we say in the introduction, an open question is to know if a similar result about non-approximability can be obtained for the b-coloring.

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