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COUNTING CLOSED AND OPEN WALKS

DELORME C / FORGE D

Unité Mixte de Recherche 8623
CNRS-Université Paris Sud – LRI

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CNRS – Université de Paris Sud
Centre d'Orsay
LABORATOIRE DE RECHERCHE EN INFORMATIQUE
Bâtiment 490
91405 ORSAY Cedex (France)

Counting closed and open walks

Charles Delorme
LRI, Charles.Delorme@lri.fr
David Forge
LRI, David.Forge@lri.fr

Abstract
We recall simple techniques for computing the numbers of Dyck or Motzkin words and similar ones, and give similar computations for some walks in 2-dimensional spaces,

Résumé
On rappelle quelques techniques pour calculer les nombres de mots de Dyck ou de Motzkin et autres *ejusdem farinae* et nous les étendons à diverses marches dans le plan.

Keywords Counting, paths

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Counting closed and open walks

C. Delorme D. Forge

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Abstract

We recall simple techniques for computing the numbers of Dyck or Motzkin words and similar ones, and give similar computations for some walks in 2-dimensional spaces,

1 Paths on the line or half-line

It is well known that the number of walks in n steps from 0 to $n-2p$ with steps of size 1 or -1 is $\binom{n}{p}$, since there must be p steps of size -1 and $n-p$ steps of size 1, and any such combination is convenient. In particular, when $n = 2p$ the number is $\binom{2p}{p}$. The corresponding generating function is $\sum_{p=0}^{\infty} \binom{2p}{p} z^{2p}$, that is $1/\sqrt{1-4z^2}$. The total number of walks is 2^n.

The ballot problem is now how many walks remain in the positive part of the line. We will call them *positive walks*. It is easily checked that the number of such walks from 0 to $n-2p$ is $\binom{n}{p}\frac{n-2p+1}{n-p+1}$. In particular when $n = 2p$, the number is $\binom{2p}{p}\frac{1}{p+1}$, a well known occurrence of the famous Catalan number. The generating function is $\sum_{p=0}^{\infty} \binom{2p}{p}\frac{1}{p+1} z^{2p}$, that is $\frac{1-\sqrt{1-4z^2}}{2z^2}$. The total number of positive walks of length n is $\binom{n}{\lfloor n/2 \rfloor}$.

In the same vein, the number of walks with steps of sizes 1 or 0 or -1 from 0 to itself give rise to the generating function $M(z) = 1/\sqrt{1-2z-3z^2}$ and the closed positive walks (Motzkin walks) produce similarly the generating function $M_+(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$ (Motzkin numbers). The total number of walks is 3^n and the positive walks give the generating function $\frac{1-zM_+(z)}{1-3z}$ (cumulated Motzkin numbers).

If instead of one step of size 0 we use two such steps, we find again Catalan numbers, the number of closed positive walks of length n is $\binom{2n+2}{n+1}\frac{1}{n+2}$, which gives the generating function $\frac{1-2z-\sqrt{1-4z}}{2z^2}$, the number of positive walks of length n is $\binom{2n+1}{n}$ and the corresponding generating function is

$\left(\frac{1}{1-4z}-1\right)/2z$. The number of closed paths in $\binom{2n}{n}$, that corresponds to the generating function $\frac{1}{\sqrt{1-4z}}$.

Instead of steps of length 1 and -1, we can use steps of size 1 and $1-k$, with $k \geq 2$ Then the number of ways to go in n steps from 0 to $n-kp$ is $\binom{n}{p}$ (with $n-p$ steps of size 1 and p steps of size $1-k$) and the number of positive walks in n steps from 0 to $n-kp$ is $f(n,p,k)=\binom{n}{p}\frac{n-kp+1}{n-p+1}$.

Indeed it is easily checked that $\binom{n}{p}\frac{n-kp+1}{n-p+1}=\binom{n-1}{p}\frac{n-kp}{n-k}+\binom{n-1}{p-1}\frac{n-kp+k}{n-p+1}$, and $\binom{n}{p}\frac{n-kp+1}{n-p+1}$ is conveniently null when $kp=n+1$.

In particular, when $n=kp$, we have $f(kp,p,k)=\binom{kp}{p}\frac{1}{kp-p+1}$.

On the other hand, the generating function $\Phi(k,z)=\sum_{p=0}^{\infty} f(kp,p,k)z^{kp}$ associated to the number of closed positive walks satisfies $\Phi(k,z)=1+z^k(\Phi(k,z))^k$.

These facts can be found in [4, ch. 7.5].

Another point of view gives the construction of closed positive walks as a language S made from two letters, say u for up and d for down, following the rule $S=\varepsilon+SuSdS\ldots dS$ with k occurrences of d..

In the same vein, the number of walks with steps of sizes 1 or 0 or -1 from 0 to itself give rise generating function $M(z)$ and the positive walks (Motzkin walks) have generating function $M_+(z)$.

Indeed, looking at the first passage at 0 after start gives $M_+(z)=1+zM_+(z)+z^2(M_+(z))^2$ and $M(z)=1+zM(z)+2z^2M(z)M_+(z)$. These relations give $M_+(z)=\frac{1-x-\sqrt{1-2z-3z^2}}{2z^2}$ and then $M(z)=1/\sqrt{1-2z-3z^2}$.

The corresponding language setting would be $S_+=\varepsilon\cup hS_+\cup uS_{\cup}dS$ and $S_-=\varepsilon\cup hS_-\cup dS_-dS_-$ and $S=\varepsilon\cup hS\cup uS_+dS\cup dS_-uS$.

More on this in the papers [1] and [3].

It is not difficult to see then that the positive walks give the generating function $\frac{\Phi(k,z)}{1-z\Phi(k,z)}$ (and more precisely the number of positive walks from the origin to the point t has generating function $z^t(\Phi(k,z))^{t+1}$) and the negative walks (or the positive walks ending at 0) give the generating function $\frac{1-z\Phi(k,z)}{1-2z}$

We now present similar results in the plane or higher dimensional spaces, and cones with summit at the origin.

Some cases where both ends on the walks are not the origin can lead to reasonably easy computations.

2 The positive line

We can look at the infinite matrix M where M_{ij} (i and j non-ngative integers) shows the number of ways to connect the origin to the point i with j steps inside the positive half-line..

It is clear that $M_{ij} = 0$ if $i - j$ is even or positive. We have said above that $M_{j-2c,j} = \binom{j}{c}\frac{j-2c+1}{j-c+1}$. Hence M is invertible and satisfies $(R + R^t)M = MR$ where R is the matrix with $R_{i,i-1} = 1$ for $i \geq 1$ and $R_{ij} = 0$ otherwise,a nd R^t is the transpose of R.

Thus the inverse N of M satisfies $RN = N(R+R^t)$ and $N_{ii} = 1$, $N_{ij} = 0$ if $i > j$. Therefore $N_{i,j} = 0$ if $i - j$ is odd and $N_{i-c,i+c} = (-1)^c\binom{i}{c}$.

An interpretation is thus: the columns of N represent the base of the second kind Chebyshev polynomials $[1, X, \ldots, U_i \ldots]$ in the base $[1, X, X^2, \ldots, X^i, \ldots]$ of the vector space of polynomials. We recall their dfinition $U_0 = 1, U_1 = X, U_{n+2} = XU_{n+1} - U_n$ or equivalently $U(2\cos\theta)\sin\theta = \sin((i+1)\theta)$ for each θ real or complex (therefore $U_i(2) = i + 1$).

If Q commutes with R, we see that $MRQ = MQR = (R + R^t)MQ$. In particular, if $Q = U_i(R)$, the matrix $MU_i(R)$ has its first column with a 1 in position i and zeros elsewhere. Thus the matrix MQ shows in its column j the number of ways to go from the point i in j steps to the points $0, 1, 2, \ldots$.

The asymptotic behaviour of the total number of positive walks of length n from the origin namely $\binom{n}{\lfloor n/2 \rfloor}$ is $2^{n+1}/(\pi\sqrt{n})$. Hence the total number of positive walks of length n from the point b is $U_i(2)2^{n+1}/(\pi\sqrt{n})$, that is $(i + 1)2^{n+1}/(\pi\sqrt{n})$,

Similarly, in the case of steps 1 and $-k$, the matrix M giving the number of ways to go from the origin to point i in j steps has $M_{j-3c,j} = \binom{j}{c}\frac{j-ck+1}{j-c+1}$ for $0 \leq c \leq j/3$ and the other entries are 0. Its inverse N has $N_{i-2c,i+c} = (-1)^c\binom{i}{c}$ for $0 \leq 2c \leq i$ and its other entries are 0.

3 Paths in the plane

We will have several kinds of steps, that are non-colinear vectors in Z^2 such that a linear combination with *positive integer* coefficients is null exists; this ensures the existence of closed walks. We may suppose without loss of generality that each vector appears in such a combination, since vectors absent from all combinations cannot be steps in a closed walk. We consider walks starting from the origin using these steps, so that the walker remains in some sector of the plan (not too small, so that it contains some steps from the origin and some steps towards the origin, in order to allow closed

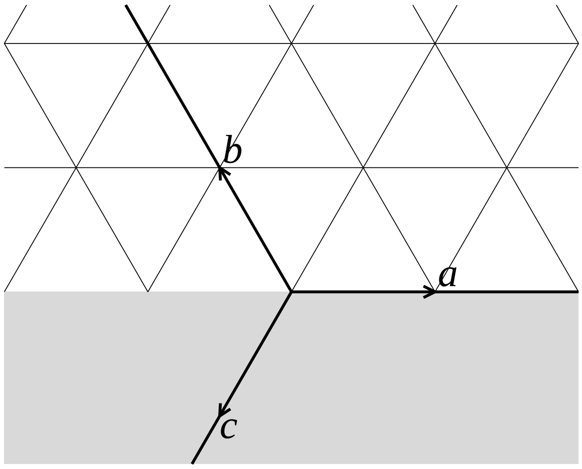

Figure 1: First half plane

walks).

3.1 The oriented triangular net

In what follows the combination is simply $a+b+c=0$ We will count walks using these steps.

3.1.1 Whole plane

Clearly, the number of walks from the origin to itself in $3p$ steps is $\frac{(3p)!}{(p!)^3}$. The corresponding generating function is $\sum_{p=0}^{\infty}\frac{(3p)!}{(p!)^3}z^{3p}$. The total number of walks of length p is of course 3^p.

3.1.2 First half-plane

We now allow only the half plane limited by the line Ra and containing b. (see figure 1). Then the closed walks inside that half plane are obtained as words with $2p$ letters from the alphabet b, c with no more c's than b's in their beginning (there are $\binom{2p}{p}\frac{1}{p+1}$ such words), with the same number of a's inserted in any possible way. Thus the number of closed walks of length $3p$ starting at the origin is $\binom{3p}{p}\binom{2p}{p}\frac{1}{p+1}=\frac{(3p)!}{(p!)^3(p+1)}$.

The number of walks arriving on the line is the number of Motzkin walks, with generating function $\frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$ and the total number of walks has generating function $1/\sqrt{1-2z-3z^2}$.

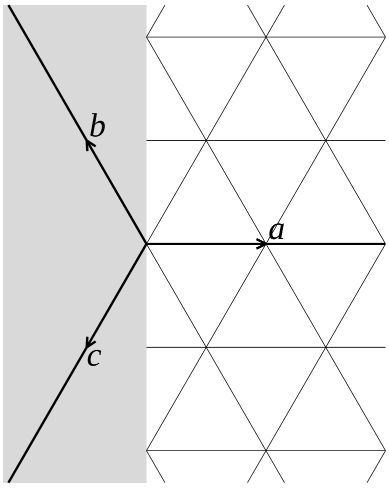

Figure 2: Second half plane

3.1.3 Second half plane

We now allow only the half plane limited by the line $R(b-c)$ and containing a (see figure 2).

Then the closed walks are obtained as words more a's than b's and c's at thir beginning; (regarding the distance to the borderline, a may be seen as a step up of length 2 and b and c as steps down of length 1), with any repartition of b and c (in equal number for a close walk). Thus the number of closed walks of length $3p$ starting at the origin is $\binom{2p}{p}\binom{3p}{p}\frac{1}{2p+1} = \frac{(3p)!}{(p!)^3(2p+1)}$.

The total number of walks arriving on the line has thus a generating function f satisfying $f = 1 + 4z^3f^3$, thus the number of walks of length $3p$ arriving on the line is $\frac{4^p\binom{3p}{p}}{2p+1}$. The number of walks on the side of a has generating function $\frac{1-2zf}{1-3z}$ and the total number on the side of b and c has generating function $\frac{f}{1-2zf}$

3.1.4 A 60-degrees sector

We now allow only the small sector between the half-line R^+a and the half-line R^-c (see figure 3).. Then the numbers x, y, z of steps a, b, c in a walk must satisfy $x \geq y \geq z$ in order that the walk starting at 0 has all its points in that sector. The number of allowed closed walk starting at the origin is $\frac{2(3p)!}{p!(p+1)!(p+2)!}$.

Indeed the number of walks of length $p+q+r$ inside that sector to go

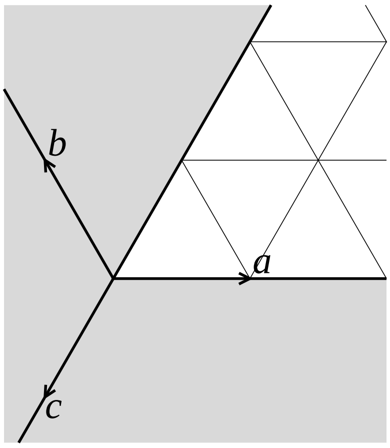

Figure 3: A 60 degrees sector

from the origin to the point M with $\overrightarrow{OM} = (p-r)a + (q-r)b$ (with $p \geq q \geq r$) is $f(p,q,r) = \frac{(p+q+r)!}{(p+2)!(q+1)!r!)(p-q+1)(p-r+2)(q-r+1)}$, since this satisfies $f(p,q,r) = f(p,q,r-1) + f(p,q-1,r) + f(p-1,q,r)$ for $p \geq q \geq r \geq 0$ $f(0,0,0) = 1$ and $f(p,q,r) = 0$ if $p = q-1$ or $q = r-1$.

The total number of walks is the number of Motzkin walks.

Indeed, we will show that the number of paths with $p \leq q \leq r$ and $p+q+r = n$ is $\binom{n}{2q}\binom{2q}{q}\frac{1}{q+1}$, in other words the number of Motzkin path with q up and q own steps (with $0 \leq 2q \leq n$.

We thus have to check the equality

$$\sum_{p=0}^{\min(q,n-2q} \frac{n!(q-p+1)(n-p-2q+1)(n-q-2p+2)}{p!(q+1)!(n-p-q+2)!} = \binom{n}{2q}\binom{2q}{q}\frac{1}{q+1}$$

that reduces to the equality $\sum_{p=0}^{\min(A,B)} \binom{A+B}{p}(A-p)(B-p)(A+B-2p) = AB\binom{A+B}{B}$, after cancellation of $n!$ and $(q+1)!$ and setting $A = q+1$ and $B = n-2q+1$ and multiplication by $(n-q+2)!$. This last equality can be proven by induction on A (tedious calculations).

There is also a bijective proof.

To a word of $\{abc\}*$ with every first factor containing more a's than b's and more b's than c's, is associated a Motzkin word by a bijection. Here is the algorithm, written in Caml (figure 4). The number of b's is also the number of pairs of parentheses.

The converse bijection is more intricated.

```
type e = A | B | C;;
type mm = U | H | D;;

let rec aux n=
  match n with
    []  ->([],0,0)
   |A::r->let (om,ob,oc)=aux r in
        if ob=0 then (H::om,0,oc) else (U::om,ob-1,oc)
   |B::r->let (om,ob,oc)=aux r in
        if oc=0 then (D::om,ob+1,0) else (H::om,ob+1,oc-1)
   |C::r->let (om,ob,oc)=aux r in (D::om,ob,oc+1);;

let abcm n = match aux n with
 (m,0,0) ->(m,"ok")
|(_,_,0) ->([], "too many B")
|(_,_,_) ->([], "too many C");;
```

Figure 4: Conversion of a word in a Motzkin word

3.1.5 A 120-degrees sector

It *seems* that the number of closed walks of length $3i$ from the origin in the small sector between the half lines R^+a and R^+b is $\frac{4^i(3i)!}{(2i+1)!(i+1)!}$.

3.1.6 A 90-degrees sector

The number of closed walk of lengths $3i$ from the origin in the small sector between the half lines R^+a and $R^+(b-c)$ is $\frac{(3i)!}{(i!)^3(2i+1)(i+1)}$. Indeed, the number of paths towards the origin in that sector using p steps a, q steps b and r steps c with $c \geq b$ and $c+b \geq 2a$ is $\frac{(a+b+c)!}{a!b!c!}\frac{b+c-2a+1}{b+c+1}\frac{c-b+1}{c+1}$.

3.2 The non-oriented triangular net

We have 3 vectors a, b, c with null sum and their opposites. The total number of walks of length n with no constraints is obviously 6^n.

Choosing the basis a, b, and coordinates x, y, one can see that the number of walks of length n with the constraint $x \geq 0$ is $2^n M_n$ where M_n is the cumulated Motzkin number (since each step $+1$, 0, -1 relative to the distance of the line $x = 0$) is given by 2 steps in the set $a, -c$ for 1, $b, -b$ for 0 and $-a, c$ for -1.

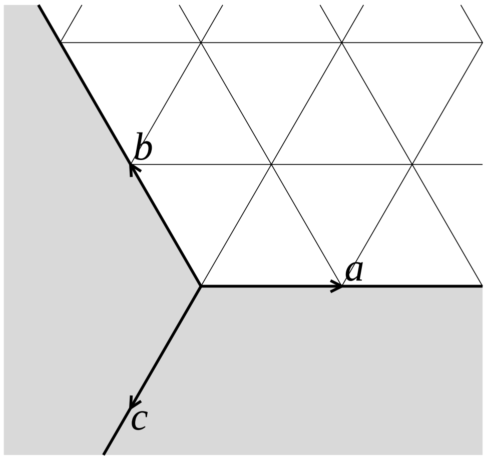

Figure 5: A 120 degrees sector

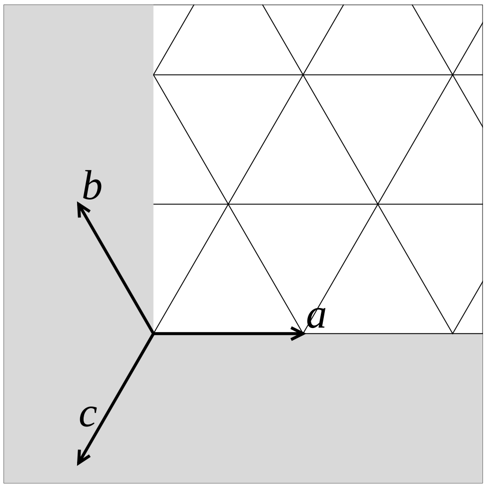

Figure 6: A 90 degrees sector

The total number of length n with the constraint $x \geq y \geq 0$ is $2^n m_n$ where m_n is the Motzkin number.

Indeed, we will use the basis $a, -c$, the sector is defined by $x >= 0 \wedge y >= 0$ and the symmetry $s : (x, y) \mapsto (y, x)$. The union of the out-neighbourhood $N(x, y)$ and of $s(N(s(x, y))$ is juste twice (as a multiset) the union of the oriented out-neighbourhoods of $N^+(x, y)$ and $s(N^+(s(x, y)$.

3.3 The square net

Here we have 4 kinds of steps, say a, $-a$, b and $-b$, with a and b independent.

3.3.1 Whole plane

The number of walks of length n is 4^n and the number of closed walks of length $2n$ is then $\binom{2n}{n}^2$. Indeed the number of walks of length t from the origin to a, b with $t - |a| - |b|$ positive and even, is $\binom{t}{t-a+b)/2}\binom{t}{(t-a-b)/2}$. It can be seen either by noticing that this expression satisfies the recurrence relation $f(a, b, t+1) = f(a-1, b, t) + f(a+1, b, t) + f(a, b-1, t) + f(a, b+1, t)$ for $t \geq 0$ and $f(0, 0, 0) = 1$ but also by noticing that the coordinates of our steps in the basis $(a + b)/2, (a - b)/2$ are ± 1 and ± 1. That last remark (taken from [2, chap. 7]) provides also the two following results.

3.3.2 First half-plane and quadrant

The number of closed walks of length $2n$ in the half-plane delimited by the line $R(b+a)$ (see figure 7) is $\binom{2n}{n}^2 \frac{1}{n+1}$. Indeed the number of walks of length t from the origin to x, y with $t - |x| - |y|$ positive and even and $x + y \geq 0$ and even is $\binom{t}{t-x+y)/2}\binom{t}{(t-x-y)/2}\frac{1}{t-x-y+1}$. The number of walks from the origin in the half-plane is $2^n \binom{n}{\lfloor n/2 \rfloor}$

The number of closed walks of length $2n$ in the quadrant $x \geq |y|$ (figure 8) is $\binom{2n}{n}^2 \frac{1}{(n+1)^2}$, The number of walks of length n from the origin is $\left(\binom{n}{\lfloor n/2 \rfloor}\right)^2$.

The product structure allows to compute also the number of paths of length n in the quadrant from a point (x, y) as the product $m(x+y, n)m(x-y, n)$ where $m(x, n)$ is the number of paths of length n from x on the positive line.

Similarly, the number of paths of length n in the half-plane $y \geq x$ from a point (x, y) is the product $m(y - x, n)2^n$.

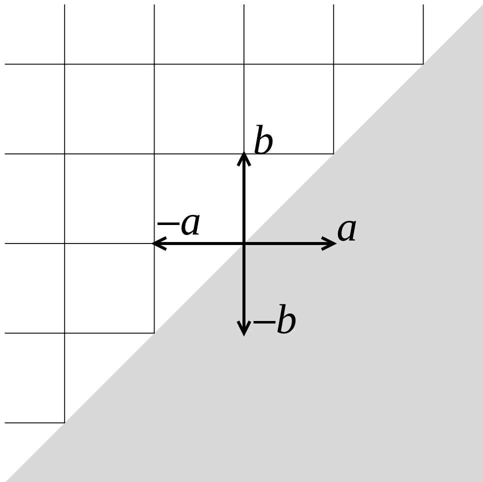

Figure 7: First half plane

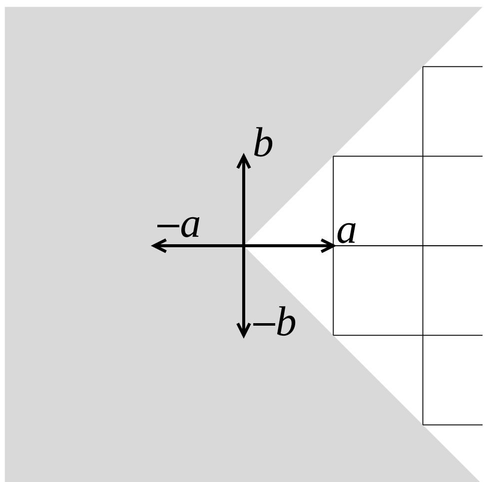

Figure 8: A quadrant

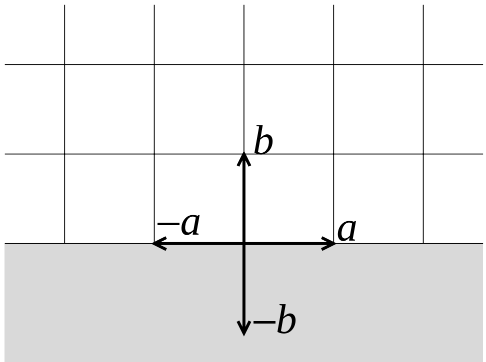

Figure 9: Another half-plane

3.3.3 Second half-plane and quadrant

Now the first expression $\binom{2n}{n}^2$ is also $\sum k = 0^m \binom{2k}{k}\binom{2m-2k}{m-k}\binom{2m}{2k}$ since we can sort the walks according to the number k (between 0 and n) of a steps. Then there are also k steps $-a$ and $m-k$ steps b and $m-k$ steps $-b$.

Thus the exponential generating function $f = \sum \binom{2n}{n}\frac{z^{2n}}{(2n)!}$ satisfies $f^2 = \sum \binom{2n}{n}\frac{1}{(n!)^2}z^{2n}$. The number of closed walks of length $2n$ in the half-plane delimited by the line Ra (see figure 9) is $K_n = \sum k = 0^n \binom{2k}{k}\frac{1}{k+1}\binom{2n-2k}{n-k}\binom{2n}{2k}$. The exponential generating function $g = \sum \binom{2k}{k}\frac{1}{k+1}\frac{z^{2k}}{(2k)!}$ clearly satisfies $2gz = f'$. Hence the exponential generating function we look at, namely $\sum K_n \frac{z^{2n}}{(2n)!}$, that is gf, satisfies $4zgf = 2ff' = (f^2)'$. This gives the wanted number as $\binom{2n}{n}^2 \frac{2n+1}{(n+1)^2}$ or $\binom{2n}{n}\frac{1}{n+1}\binom{2n+2}{n+1}\frac{1}{2}$.

Similarly, since $(z^2g)' = 2zf$, we have $zg' + 2g = 2f$, and $(z^4g^2)' = 2z^4gg' + 4z^3g^2 = 4z^3fg$. Thus the coefficient of z^{2n} in g^2 is $\binom{2n+2}{n+1}\frac{1}{n!(n+2)!}$ and the number of closed walks of length $2n$ in the quadrant $a \geq 0 \wedge b \geq 0$ (see figure 10) is $\binom{2n}{n}\frac{1}{n+1}\binom{2n+2}{n+1}\frac{1}{n+2}$. The above results can also be obtained (like in [2]) by Vandermonde convolution.

Indeed

$$\begin{aligned}\sum_{k=0}^{n}\binom{2n}{2k}\frac{\binom{2k}{k}}{k+1}\binom{2n-2k}{n-k} &= \sum_{k=0}^{n}\frac{(2n)!}{(2n-2k)!(2k)!}\frac{(2k)!}{k!(k+1)!}\frac{(2n-2k)!}{(n-k)!(n-k)!}\\ &= \sum_{k=0}^{n}\frac{(2n)!}{n!(n+1)!}\frac{n!}{k!(n-k)!}\frac{(n+1)!}{(k+1)!(n-k)!}\\ &= \frac{(2n)!}{n!(n+1)!}\sum_{k=0}^{n}\binom{n}{k}\binom{n+1}{n-k}\end{aligned}$$

and Vandermonde convolution gives then $\sum_{k=0}^{n}\binom{n}{k}\binom{n+1}{n-k} = \binom{2n+1}{n}$. Now the result $\frac{(2n)!}{n!(n+1)!}\binom{2n+1}{n}$ equals the formerly obtained formula $\binom{2n}{n}\frac{1}{n+1}\binom{2n+2}{n+1}\frac{1}{2}$.

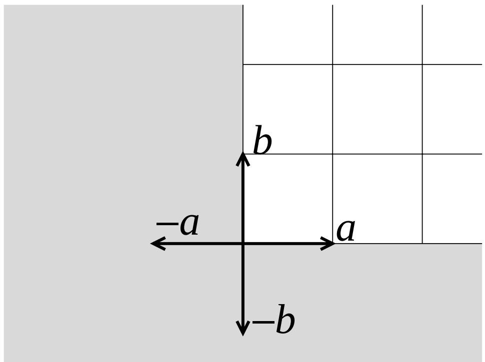

Figure 10: Another quadrant

And in the same way $\sum_{k=0}^{n} \binom{2n}{2k} \frac{\binom{2k}{k}}{k+1} \frac{\binom{2n-2k}{n-k}}{n-k+1}$ an be rewritten as the product $\frac{(2n)!}{(n+1)!(n+1)!} \sum_{k=0}^{n} \binom{n+1}{k} \binom{n+1}{n-k}$ where the sum simplifies to $\binom{2n+2}{n}$.

The number of walks in the half-plane is $\binom{2n+1}{n}$. This can be obtained by noticing that the number of walks arriving on the line gives a generating function ϕ that satisfies $\phi = 1 + 2z\phi + \phi^2$, thus $\phi = \frac{1-2z-\sqrt{1-4z}}{z^2}$ and the total number of walks gives the generating function $\frac{1-z\phi}{1-4z}$.

The number of walks in the quadrant is $\binom{n}{\lfloor n/2 \rfloor}\binom{n+1}{\lfloor (n+1)/2 \rfloor}$. Indeed the exponential generating function for positive walks on the line is $\sum \binom{n}{\lfloor n/2 \rfloor} z^n = f + \frac{f'}{2} = f + \frac{g}{2z}$, the exponential generating function for the walks from the origin in the quadrant is its square, and we have already computed the coefficients for f^2, fg and g^2.

4 Higher dimensions

Here we consider in the d-dimensional space a set of $d+1$ vectors $e_i, i = 0..d$ with only one relation, namely $\sum e_i = 0$. The number of closed walks of length dn from the origin to itself is then $\frac{(dn+n)!}{(n!)^{d+1}}$.

The number of walks in a cone is easily computed when the cone is described by a list of inequalities of the following kind:

- the number of steps p_x counting the steps e_x and p_y counting the steps e_x with $x < y$ must satisfy $p_x \le p_y$ along the walk. Indeed this number of closed walks is $\frac{(nd+n)!}{n!(n+1)!\cdots(n+d)!}$ and the number of walks from the origin to $\sum_{i=0}^{d} p_i e_i$ of length $\sum_{i=0}^{d} p_i$ is then $\frac{(\sum p_i)!}{\prod (p_i+i)!} \prod_{i<j} (p_j - p_i + j - i)$.

- the inequality corresponds to a half-space, say delimited by the hyperplane of equation $de_0 = e_1+e_2+\cdots+e_d$, thus according a privileged role to e_0. Then the number of closed paths from the origin is $\frac{(dn+n)!}{(n!)^{d+1}(dn+1)}$

As a corollary, it is easy to compute the number of walks if the inequalities are described (up to a permutation of the indices) as follows: the set of steps is partitioned into subsets with either

- a total order, and the corresponding p_i's satisfy $p_i \leq p_j$ if i, j belong to the same part, and $i < j$ for the order of that part, or

- a privilegied generator as above

but nothing is required if i and j are not in the same part.

For example, with $d = 3$, if the orders are $0 < 1$ and $2 < 3$, in other words the constraints read $(p_0 \leq p1) \wedge (p_2 \leq p_3)$, the number of closed walks of length $4i$ starting at the origin is $\frac{(4*i)!}{(i!(i+1)!)^2}$.

Some other sectors of the space give simple formulas. For example, using 4 steps with null sum in 3-dimensional space, the constraints for half-spaces $2p_0 \leq p_2 + p_3$ and $p_0 + p_1 \leq p_2 + p_3$ both give for the number of closed walks of length $4n$ the expression $\frac{(4n)!}{(n!)^4(2n+1)}$.

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