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ALGORITHM FOR TWO DISJOINT LONG PATHS IN CONNECTED GRAPHS

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Algorithm for two disjoint long paths in connected graphs*

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Abstract

Denote by $\bar{\sigma}_k = \min\,\{d(x_1)+d(x_2)+\cdots+d(x_k)-|N(x_1)\cap N(x_2)\cap\cdots\cap N(x_k)| \mid x_1, x_2, \cdots, x_k$ are k independent vertices in $G\}$. Let n and m denote the number of vertices and edges of G. For any connected graph G, we give a polynomial algorithm in $O(nm)$ time to either find two disjoint paths P_1 and P_2 such that $|P_1| + |P_2| \geq min\{\bar{\sigma}_4, n\}$ or output $G = \cup_{i=1}^{k} G_i$ such that for any $i, j \in \{1, 2, ..., k\}$ $(k \geq 3)$, $V(G_i) \cap V(G_j) = \{v\}$, where $v \in V(G)$.

Keywords: path, degree sum, dominating path

1 Introduction and Notations

In this paper we consider finite graphs without loops or multiple edges. We use [2] for terminology and notations not defined here. Let n and m denote the number of vertices and edges of G. A hamiltonian cycle (path, resp.) is a spanning cycle (path, resp.) of the graph. A graph G is called hamiltonian if G has a hamiltonian cycle. The circumference $c(G)$ of graph G is the longest cycle in graph. Given a subset $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. Let $P = x_1x_2...x_p$ be a path in graph G. We use $P[x_i, x_j]$ or x_iPx_j to denote the sub-path $x_ix_{i+1}...x_j$ of P. Define $P(x_i, x_j] = P[x_{i+1}, x_j]$, $P[x_i, x_j) = P[x_i, x_{j-1}]$ and $P(x_i, x_j) = P[x_{i+1}, x_{j-1}]$. We use similar definitions for a cycle. For any i, we put $x_i^+ = x_{i+1}$, $x_i^- = x_{i-1}$, $x_i^{+2} = x_{i+2}$ and $x_i^{-2} = x_{i-2}$. For a vertex set $A \subseteq P$, $A^+ = \{x^+ \mid x \in A\}$, $A^- = \{x^- \mid x \in A\}$, $A^{+2} = (A^+)^+$ and $A^{-2} = (A^-)^-$. For a vertex x of G, a neighbor of x means a vertex adjacent to x, denoted by $N_G(x)$, and the

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degree of x is the number of neighbors of x, denoted by $d(x)$. Let $N_P(x)^{-j} = \{x_i \mid x_i^{+j} \in N_P(x)\}$, $j \geq 1$. A path P is called *dominating* if no component of $G - P$ has more than one vertex. Let $\bar{\sigma}_k = \min\ \{d(x_1) + d(x_2) + \cdots + d(x_k) - |N(x_1) \cap N(x_2) \cap \cdots \cap N(x_k)| \mid x_1, x_2, \cdots, x_k$ are k independent vertices in $G\}$ and $\sigma_k = \min\ \{d(x_1) + d(x_2) + \cdots + d(x_k) \mid x_1, x_2, \cdots, x_k$ are k independent vertices in $G\}$.

Various long path and cycle problems are interesting and important in graph theory and have been deeply studied. Two classical results are due to Dirac and Ore respectively.

Theorem 1.1 (Dirac [3]) *Let G be a graph on $n \geq 3$ vertices. If the minimum degree $\delta \geq \frac{n}{2}$, G is hamiltonian.*

Theorem 1.2 (Ore [7]) *Let G be a graph on $n \geq 3$ vertices. If $\sigma_2 \geq n$, G is hamiltonian.*

It is natural to consider sufficient conditions concerning the degree sum of more independent vertices. Flandrin, Jung and Li [4] investigated the degree sum of three independent vertices and obtained the following result.

Theorem 1.3 (Flandrin, Jung and Li [4]) *Let G be a 2-connected graph of order n. If $\bar{\sigma}_3 \geq n$, G is hamiltonian.*

These results are also generalized to the circumferences of the graphs.

Theorem 1.4 (Dirac [3]) *Let G be a 2-connected graph on $n \geq 3$ vertices. Then $c(G) \geq min\{n, 2\delta\}$.*

Theorem 1.5 (Bermond [1]) *Let G be a 2-connected graph on $n \geq 3$ vertices. Then $c(G) \geq min\{n, \sigma_2\}$.*

Theorem 1.6 (Wei [8]) *Let G be a 3-connected graph on $n \geq 3$ vertices. Then $c(G) \geq min\{n, \bar{\sigma}_3\}$.*

H. Li [6] further studied the degree sum of four independent vertices in 3-connected graphs and proved:

Theorem 1.7 (Li [6]) *Let G be a 3-connected graph of order n. If $\bar{\sigma}_4 \geq n + 3$, G has a dominating maximum cycle.*

Moreover, Zhang and Li [9] gave a bound of the length of a path by the neighborhood condition of any three independent vertices of the path.

Theorem 1.8 (Zhang and Li [9]) *Let G be a 2-connected graph of order $n \geq 3$. Then there exists a vertex x and a path P such that x is an end-vertex of P and P contains at least $min\{n, \Gamma_3(x, P) + 1\}$ vertices. Furthermore, P can be found in $O(nm)$ time.*

This paper investigates four independent vertices in graph G. The main result is the following:

Theorem 1.9 *Let G be a connected graph. Then G has two disjoint paths P_1 and P_2 satisfying $|P_1| + |P_2| \geq min\{\bar{\sigma}_4, n\}$ or $G = \cup_{i=1}^{k} G_i$ such that for any $i, j \in \{1, 2, ..., k\}$ $(k \geq 3)$, $V(G_i) \cap V(G_j) = \{v\}$, where $v \in V(G)$.*

In the following two sections, we show that finding two disjoint paths can be realized by a polynomial algorithm. Such an algorithm with time complexity $O(mn)$ is given in this paper.

2 Algorithm

Let $P_1 = u_0u_2...u_p$ and $P_2 = v_0v_2...v_q$ be two paths satisfying:

(a) $P_1 \cup P_2$ covers as many vertices as possible,

(b) subject to (a), P_1 is as long as possible,

(c) subject to (a) and (b), P_2 is as long as possible.

Based on (a), (b) and (c), two paths P_1 and P_2 are constructed such that $|P_1| + |P_2| \geq min\{\bar{\sigma}_4, n\}$.

Circumstance 1: There is a vertex $v \in V(G)\backslash V(P_1)$ which is adjacent to one end-vertex of P_1.

Operation 1: Extend P_1 by adding v.

Figure 1.

Circumstance 2: u_0 is adjacent to u_p and $V(G)\backslash V(P_1) \neq \emptyset$.

Operation 2: Let v be a vertex in $V(G)\backslash V(P_1)$ which is adjacent to a vertex u_i of P_1. Reset $P_1 = vu_iu_{i-1}...u_0u_pu_{p-1}...u_{i+1}$.

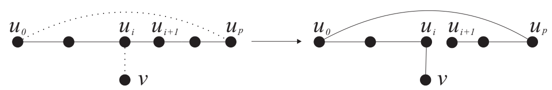

Figure 2.

Circumstance 3: $u_i \in N_{P_1}(u_0) \cap N_{P_1}(u_p)^+$ and $V(G)\backslash V(P_1) \neq \emptyset$.

Operation 3: Reset $P_1 = u_iu_{i+1}...u_pu_{i-1}u_{i-2}...u_0$ and then extend it further by operation 2.

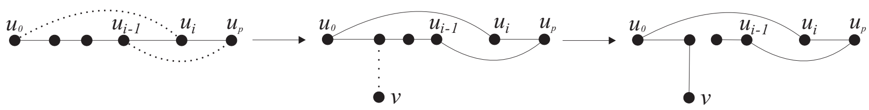

Figure 3.

Circumstance 4: There is a vertex $v \in V(G)\backslash V(P_1)$ such that $u_i, u_{i+1} \in N_{P_1}(v)$.

Operation 4: Reset $P_1 = u_0...u_i v u_{i+1}...u_p$.

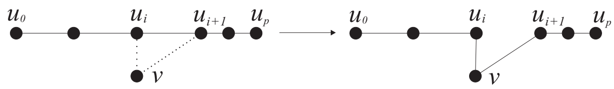

Figure 4.

Circumstance 5: There is a vertex $v^{'} \in V(G)\backslash V(P_1) \cup V(P_2)$ which is adjacent to one end-vertex of P_2.

Operation 5: Extend P_2 by adding $v^{'}$.

Circumstance 6: v_0 is adjacent to v_q and $V(G)\backslash V(P_1) \cup V(P_2) \neq \emptyset$.

Operation 6: Let $v^{'}$ be a vertex in $V(G)\backslash V(P_1) \cup V(P_2)$ which is adjacent to a vertex v_i of P_2. Reset $P_2 = v^{'} v_i v_{i-1}...v_0 v_q v_{q-1}...v_{i+1}$.

Circumstance 7: $v_i \in N_{P_2}(v_0) \cap N_{P_2}(v_q)^+$ and $V(G)\backslash V(P_1) \cup V(P_2) \neq \emptyset$.

Operation 7: Reset $P_2 = v_i v_{i+1}...v_q v_{i-1} v_{i-2}...v_0$ and extend it further by operation 6.

Circumstance 8: There is a vertex $v^{'} \in V(G)\backslash V(P_1) \cup V(P_2)$ such that $v_i, v_{i+1} \in N_{P_2}(v^{'})$.

Operation 8: Reset $P_2 = v_0...v_i v^{'} v_{i+1}...v_q$.

Circumstance 9: There exists a vertex $u_i \in N_{P_1}(v_0)^{+j} \cap N_{P_1}(u_0)$, $1 \leq j \leq |P_2|$.

Operation 9: Reset $P_1 = v_q...v_0 u_i^{-j} u_i^{-(j+1)}...u_0 u_i u_{i+1}...u_p$.

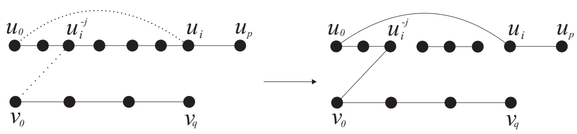

Figure 5.

Circumstance 10: There exists a vertex $u_i \in N_{P_1}(v_0)^{-j} \cap N_{P_1}(u_p)$, $1 \leq j \leq |P_2|$.

Operation 10: Reset $P_1 = v_q...v_0 u_i^{+j} u_i^{+(j+1)}...u_p u_i u_{i-1}...u_0$.

Circumstance 11: There exist two different vertices $u_i \in N_{P_1}(v_0)^{+l} \cap N_{P_1}(u_p)$ and $u_j \in N_{P_1}(v_0)^{-m} \cap N_{P_1}(u_0)$, $i < j$, $1 \leq min\{m, l\} \leq |P_2|$.

Operation 11: If $min\{m, l\} = m$, reset $P_1 = v_q...v_0 u_j^{+m}...u_p u_i u_{i-1}...u_0 u_j u_{j-1}...u_{i+1}$. If $min\{m, l\} = l$, reset $P_1 = v_q...v_0 u_i^{-l}...u_0 u_j...u_p u_i u_{i+1}...u_{j-1}$.

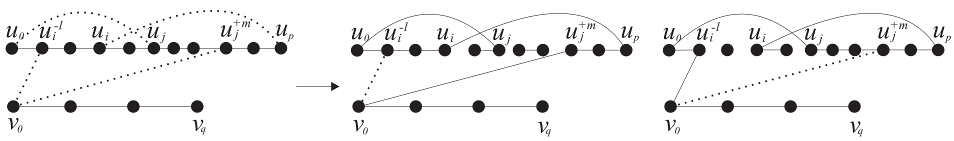

Figure 6.

Circumstance 12: There exist two vertices $u_k \in N_{P_1}(v_0)$ and $u_i \in N_{P_1}(u_0)^{-j} \cap N_{P_1}(u_p)$, $1 \leq j \leq |P_2|$.

Operation 12: If $k < i$ or $k > i+j$, reset $P_1 = v_q...v_0u_ku_{k-1}...u_0u_i^{+j}...u_pu_iu_{i-1}...u_{k+1}$ or $P_1 = v_q...v_0u_ku_{k+1}...u_pu_iu_{i-1}...u_0u_i^{+j}...u_{k-1}$. If $i \leq k \leq i+j$, reset $P_1 = v_q...v_0u_ku_{k-1}...u_0$ $u_i^{+j}...u_p$ or $P_1 = v_q...v_0u_ku_{k+1}...u_pu_iu_{i-1}...u_0$

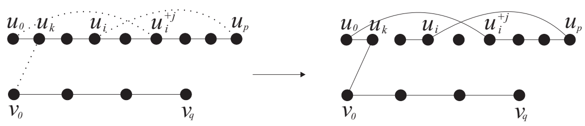

Figure 7.

Similarly for vertex v_q, repeat operations 9 to 12.

Circumstance 13: There exists a vertex $v'' \in V(G) \backslash V(P_1) \cup V(P_2) \cup V(P_3)$ which is adjacent to one end-vertex of P_3.

Repeat the same operations 1, 2, 3 and 4, until such operations can no longer be carried out. Set $P_3 = w_0w_1...w_l$.

Circumstance 14: There exists a vertex $u_i \in N_{P_1}(w_0)^{+j} \cap N_{P_1}(u_0)$, $1 \leq j \leq |P_3|$.

Operation 14: Reset $P_1 = w_l...w_0u_i^{-j}...u_0u_iu_{i+1}...u_p$.

Circumstance 15: There exists a vertex $u_i \in N_{P_1}(w_0)^{-j} \cap N_{P_1}(u_p)$, $1 \leq j \leq |P_3|$.

Operation 15: Reset $P_1 = w_l...w_0u_i^{+j}...u_pu_iu_{i-1}...u_0$.

Circumstance 16: There exists a vertex $u_i \in N_{P_1}(w_0)^{-j} \cap N_{P_1}(v_0)$ or $u_i \in N_{P_1}(w_0)^{+j} \cap N_{P_1}(v_0)$, $1 \leq j \leq |P_3|$.

Operation 16: Reset $P_1 = v_q...v_0u_iu_{i-1}...u_0$ and $P_2 = w_l...w_0u_i^{+j}...u_p$ such that $|P_1| \geq |P_2|$.

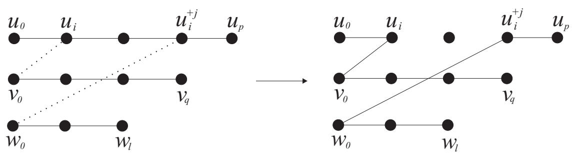

Figure 8.

Circumstance 17: There exist two vertices $u_i \in N_{P_1}(v_0) \cap N_{P_1}^{-m}(u_p)$ and $u_j \in N_{P_1}(w_0)$, $j > i + m$, $1 \leq m \leq |P_3|$.

Operation 17: Reset $P_1 = v_q...v_0u_iu_{i-1}...u_0$ and $P_2 = w_l...w_0u_j...u_pu_i^{+m}...u_{j-1}$ such that $|P_1| \geq |P_2|$.

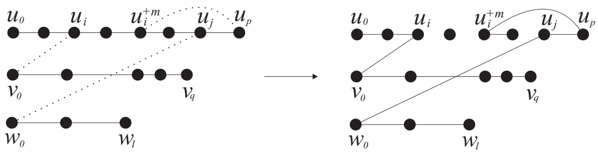

Figure 9.

Circumstance 18: There exists a vertex $v_i \in N_{P_2}(w_0)^{+j} \cap N_{P_2}(v_0)$, $1 \leq j \leq |P_3|$.

Operation 18: Reset $P_2 = w_l...w_0v_i^{-j}...v_0v_iv_{i+1}...v_q$.

Circumstance 19: There exists a vertex $v_i \in N_{P_2}(w_0)^{-j} \cap N_{P_2}(v_p)$, $1 \leq j \leq |P_3|$.

Operation 19: Reset $P_2 = w_l...w_0v_i^{+j}...v_pv_iv_{i-1}...v_0$.

Circumstance 20: There exists a vertex $v_i \in N_{P_2}(w_0) \cap \{v_{q-l+1}, ..., v_q\}$.

Operation 20: Reset $P_2 = w_l...w_0v_i...v_0$.

Similarly for vertex w_l, repeat operations 14 to 19.

Algorithm

Input: A connected graph G.

Output: Two disjoint paths P_1 and P_2 which cannot be extended by operations or $G = \cup_{i=1}^{k} G_i$ such that for any $i, j \in \{1, 2, ..., k\}$ $(k \geq 3)$, $V(G_i) \cap V(G_j) = \{v\}$, where $v \in V(G)$.

Step 1. Set $P_1 = v$, where v is an arbitrary vertex in G.

Step 2. Extend P_1 repeatedly by Operations 1 to 4 until such operations can no longer be carried out.

Step 3. If $V(G)\backslash V(P_1) = \emptyset$, set $P_2 = \emptyset$ and output P_1 and P_2, stop. Else, set $P_2 = v$, where v is an arbitrary vertex in $V(G)\backslash V(P_1)$.

Step 4. Extend P_2 repeatedly by operations 5 to 8 until such operations can no longer be carried out.

Step 5. If $V(G)\backslash V(P_1) \cup V(P_2) = \emptyset$, output P_1 and P_2; stop. Else, if one of circumstances 9 to 12 happens, extend P_1 by the corresponding operation; go to step 2. If one of circumstances 6 to 8 happens, extend P_2 by the corresponding operation; go to step 5.

Step 6. If $V(G)\backslash V(P_1) \cup V(P_2) = \emptyset$, output P_1 and P_2; stop. Else, set $P_3 = v$, where v is an arbitrary vertex in $V(G)\backslash V(P_1) \cup V(P_2)$.

Step 7. Extend P_3 by operations similarly as 1 to 4 until such operations can no longer be carried out.

Step 8. If one of circumstances 14 to 17 happens, extend P_1 by the corresponding operation; go to step 2. If circumstances 18 to 20 happens, extend P_2 by the corresponding operation; go to step 5.

Suppose that v_0 and w_0 are respective end-vertices of P_2 and P_3.

Step 9. If $N_{P_2}(w_0) \neq \emptyset$, or $|N_{P_1}(v_0)| \geq 2$, or $|N_{P_1}(w_0)| \geq 2$, output P_1 and P_2. Else, set $P_i = v$ $(i \geq 4)$, where v is an arbitrary vertex in $V(G) \backslash V(P_1) \cup V(P_2) \cup V(P_3)$.

Step 10. Extend P_i by operations similarly as 1 to 4 until such operation can no longer be carried out, and then repeat steps 8 and 9 for the end-vertex of P_i.

Step 11. If $V(G) \backslash \cup_{i=1}^{k} V(P_i) \neq \emptyset$, go to step 9. Else, output $G = \cup_{i=1}^{k} G[V(P_i)]$; stop.

To prove the main Theorem, we need the following Lemmas.

Lemma 2.1 *Let G be a graph, $P = v_1 v_2 ... v_p$ a path in G and u_1, u_2, u_3, u_4 four vertices in $V(G) - V(P)$. Suppose that $v_1 \notin N_P(u_2)$ and $v_p \notin N_P(u_2)$. If for any integer $m \geq 2$, the following hold:*

(i) $N_P(u_2)^{-j} \cap N_P(u_4) = \emptyset$, $N_P(u_1)^{-j} \cap N_P(u_2) = \emptyset$ and $N_P(u_1)^{-j} \cap N_P(u_4) = \emptyset$, $1 \leq j \leq m$,

(ii) $N_P(u_3)^{-} \cap N_P(u_4) = \emptyset$, $N_P(u_1)^{-} \cap N_P(u_3) = \emptyset$, $N_P(u_2)^{+} \cap N_P(u_2) = \emptyset$ and $N_P(u_3)^{+} \cap N_P(u_3) = \emptyset$,

(iii) $N_P(u_2)^{-} \cap N_P(u_3) = \emptyset$ and $N_P(u_3)^{-} \cap N_P(u_2) = \emptyset$,

then $\sum_{i=1}^{4} d_P(u_i) \leq p + 2 + \lambda$, where $\lambda = |\cap_{i=1}^{4} N_P(u_i)|$.

Proof. If $|P| = 1$, the result is trivial. Assume that the result holds for any path P' with $|P'| < |P|$.

Suppose that $N_P(u_1) = \{v_1\}$ and $N_P(u_4) = \{v_p\}$. If one of $v_1 \notin N_P(u_2)$ and $v_p \notin N_P(u_2)$ holds, by (iii), $\sum_{i=1}^{4} d_P(u_i) \leq p + 2 + \lambda$. So assume $N_P(u_1) \neq \{v_1\}$. Let $N_P(u_1) = \{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$. If for any two consecutive vertices $v_{i_{l-1}}$ and v_{i_l} of $N_P(u_1)$, $i_l - i_{l-1} \leq 3$ and $i_1 \leq 4$, denote $P_1 = P[v_1, v_{i_1 - 1}]$, $P_2 = P[v_{i_1}, v_{i_2 - 1}]$, $\cdots$, $P_k = P[v_{i_{k-1}}, v_{i_k - 1}]$, $P_{k+1} = P[v_{i_k}, v_p]$. By (i) and (ii), $\sum_{i=1}^{4} d_{P_j}(u_i) \leq |P_j| + \lambda_j$, $j = 1, ..., k$. By induction hypothesis, $\sum_{i=1}^{4} d_{P_{k+1}}(u_i) \leq |P_{k+1}| + 2 + \lambda_{k+1}$. Thus, $\sum_{i=1}^{4} d_P(u_i) = \sum_{j=1}^{k+1} \sum_{i=1}^{4} d_{P_j}(u_i) \leq p + 2 + \lambda$. Let v_{i_l} be the first vertex of $N_P(u_1)$ such that $i_l - i_{l-1} \geq 4$ or $i_1 \geq 5$. By (i), $v_{i_l - 1} \notin \cup_{i=1}^{4} N_P(u_i)$. If $v_{i_l - 2} \notin \cup_{i=1}^{4} N_P(u_i)$, denote $P_1 = P[v_1, v_{i_l - 3}]$ and $P_2 = P[v_{i_l}, v_p]$. Since $v_1 \notin N_{P_1}(u_2)$ and $v_p \notin N_{P_2}(u_2)$, by induction hypothesis, $\sum_{i=1}^{4} d_{P_1}(u_i) \leq |P_1| + 2 + \lambda_1$ and $\sum_{i=1}^{4} d_{P_2}(u_i) \leq |P_2| + 2 + \lambda_2$. Then $\sum_{i=1}^{4} d_P(u_i) \leq |P_1| + 2 + \lambda_1 + |P_2| + 2 + \lambda_2 = p + 2 + \lambda$. Assume $v_{i_l - 2} \in \cup_{i=1}^{4} N_P(u_i)$. By (i), $v_{i_l - 2} \in N_P(u_3)$. By (ii) and (iii), $v_{i_l - 3} \notin \cup_{i=1}^{4} N_P(u_i)$. Denote $P_1 = P[v_1, v_{i_l - 4}]$, $P_2 = v_{i_l - 2}$ and $P_3 = P[v_{i_l}, v_p]$. Then $\sum_{i=1}^{4} d_P(u_i) \leq |P_1| + 2 + \lambda_1 + |P_2| + |P_3| + 2 + \lambda_3 = p + 2 + \lambda$. □

Lemma 2.2 *Let G be a graph, $P = v_1 v_2 ... v_p$ a path in G and u_1, u_2, u_4 three vertices in $V(G) - V(P)$. If for any integer $m \geq 2$, the following hold:*

(i) $N_P(u_1)^{-j} \cap N_P(u_2) = \emptyset$ *and* $N_P(u_4)^{+j} \cap N_P(u_2) = \emptyset$, $1 \leq j \leq m$,

(ii) for two consecutive vertices v_i *and* v_j $(j > i)$ *of* $N_P(u_2)$*, either* $\{v_{j-m}, ..., v_{j-1}\} \cap (\cup_{i=1,4} N_P(u_i)) = \emptyset$ *or* $\{v_{i+1}, ..., v_{i+m}\} \cap (\cup_{i=1,4} N_P(u_i)) = \emptyset$,

(iii) $N_P(u_1)^{-} \cap N_P(u_4) = \emptyset$ *and* $N_P(u_2)^{+} \cap N_P(u_2) = \emptyset$,

then $\sum_{i=1,2,4} d_P(u_i) \leq p + 2$.

Proof. If $|P| = 1$, the result is trivial. Assume that the result holds for any path P' with $|P'| < |P|$.

Suppose $|N_P(u_2)| = 1$. By (iii), $\sum_{i=1,2,4} d_P(u_i) \leq p + 2$. So assume that $|N_P(u_2)| \geq 2$. Let v_i, v_j be the first and the second vertices of $N_P(u_2)$. Denote $P_1 = P[v_1, v_i]$, $P_2 = P[v_{i+1}, v_{j-1}]$ and $P_3 = P[v_j, v_p]$. Since v_i is the first vertex of $N_P(u_2)$, similarly, $\sum_{i=1,2,4} d_{P_1}(u_i) \leq |P_1| + 2$ and the equality holds only if $v_i \in \cap_{i=1,2,4} N_{P_1}(u_i)$. By induction hypothesis, $\sum_{i=1,2,4} d_{P_3}(u_i) \leq |P_3| + 2$. If $j - i \leq m + 1$, by (i), $\sum_{i=1,2,4} d_{P_2}(u_i) = 0$. If $\sum_{i=1,2,4} d_{P_1}(u_i) \leq |P_1| + 1$, $\sum_{i=1,2,4} d_P(u_i) = \sum_{i=1,2,4}(d_{P_1}(u_i) + d_{P_2}(u_i) + d_{P_3}(u_i)) \leq |P_1| + 1 + |P_3| + 2 = |P| - |P_2| + 3$. By (iii), $|P_2| \geq 1$. Thus $\sum_{i=1,2,4} d_P(u_i) \leq |P| + 2$. If $\sum_{i=1,2,4} d_{P_1}(u_i) = |P_1| + 2$, $v_i \in \cap_{i=1,2,4} N_P(u_i)$ and then by (i), $|P_2| \geq 2$. Hence $\sum_{i=1,2,4} d_P(u_i) \leq |P_1| + 2 + |P_3| + 2 = |P| - |P_2| + 4 \leq |P| + 2$. If $j - i \geq m + 2$, by (ii) and (iii), $\sum_{i=1,2,4} d_{P_2}(u_i) \leq |P_2| - m$. Thus $\sum_{i=1,2,4} d_P(u_i) \leq |P_1| + 2 + |P_3| + 2 + |P_2| - m = |P| + 4 - m$. As $m \geq 2$, $\sum_{i=1,2,4} d_P(u_i) \leq p + 2$. □

Lemma 2.3 *Let* G *be a graph,* $P = v_1 v_2 ... v_p$ *a path in* G *and* u_2, u_3 *two vertices in* $V(G) - V(P)$*. If for any integer* $l \geq 1$*, the following hold:*

(i) $N_P(u_3) \neq \emptyset$ *and* $N_P(u_3) \cap \{v_{p-l+1}, ..., v_p\} = \emptyset$,

(ii) $N_P(u_3)^{+} \cap N_P(u_3) = \emptyset$ *and* $N_P(u_2)^{-j} \cap N_P(u_3) = \emptyset$, $1 \leq j \leq l$,

then $\sum_{i=2,3} d_P(u_i) \leq p - l + 1$.

Proof. We proceed by induction on $|N_P(u_3)|$. If $|N_P(u_3)| = 1$, by (ii), $\sum_{i=2,3} d_P(u_i) \leq p - l + 1$.

Assume the result holds for $|N_P(u_3)| < k$. Suppose that $N_P(u_3) = \{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$. If for any consecutive vertices $v_{i_{j-1}}$ and v_{i_j} of $N_P(u_3)$, $i_j - i_{j-1} \leq l$, by (i) and (ii), $N_P(u_2) \cap \{v_{i_1+1}, v_{i_1+2}, ..., v_{i_k}, v_{i_k}^{+}, ..., v_{i_k}^{+l}\} = \emptyset$. As $N_P(u_3)^{+} \cap N_P(u_3) = \emptyset$, $\sum_{i=2,3} d_P(u_i) \leq p - l - (|N_P(u_3)| - 1) + 1 = p - l + 1 + 1 - |N_P(u_3)| \leq p - l + 1$.

So assume there exist two consecutive vertices v_{i_j} and $v_{i_{j+1}}$ of $N_P(u_3)$ such that $i_{j+1} - i_j \geq l + 1$. Denote $P_1 = P[v_1, v_{i_j+l}]$ and $P_2 = P[v_{i_j+l+1}, v_p]$. By induction hypothesis, $\sum_{i=2,3} d_{P_1}(u_i) \leq |P_1| - l + 1$ and $\sum_{i=2,3} d_{P_2}(u_i) \leq |P_2| - l + 1$. Thus $\sum_{i=2,3} d_P(u_i) = \sum_{i=2,3}(d_{P_1}(u_i) + d_{P_2}(u_i)) \leq |P_1| - l + 1 + |P_2| - l + 1 = p - l + 1 + 1 - l \leq p - l + 1$. □

3 Proof of Theorem 1.9

Since each of 20 operations either extends P_1 or increases P_2 by at least one vertex, at most $O(n)$ extensions are needs. Furthermore, each extension can be completed in $O(m)$ time by graph searching (see for example [5]). Hence, P can be found in $O(mn)$ time. In the following, we prove that G has two paths P_1 and P_2 satisfying $|P_1|+|P_2| \geq min\{\bar{\sigma}_4, n\}$ or $G = \cup_{i=1}^{k} G_i$ such that for any $i, j \in \{1, 2, ..., k\}$ $(k \geq 3)$, $V(G_i) \cap V(G_j) = \{v\}$, where $v \in V(G)$.

Without loss of generality, we assume that $P_3 \neq \emptyset$. Let $P_1 = u_0u_1...u_p$, $P_2 = v_0v_1...v_q$ and $P_3 = w_0w_1...w_l$ be the paths found by Algorithm 1. By Operations 1, 2 and 5, u_0, u_p, v_0 and w_0 are independent vertices. Furthermore,

$N_{P_1}(u_0) \cap N_{P_1}(u_p)^+ = \emptyset$ (by Operation 3),

$N_{P_1}(v_0)^+ \cap N_{P_1}(v_0) = \emptyset$ and $N_{P_1}(w_0)^+ \cap N_{P_1}(w_0) = \emptyset$ (by Operation 4),

$N_{P_2}(w_0)^+ \cap N_{P_2}(w_0) = \emptyset$ (by Operation 8),

$N_{P_1}(v_0)^{+j} \cap N_{P_1}(u_0) = \emptyset$ and $N_{P_1}(v_0)^{-j} \cap N_{P_1}(u_p) = \emptyset$, $1 \leq j \leq q$ (by Operations 9 and 10),

$N_{P_1}(w_0)^{+j} \cap N_{P_1}(u_0) = \emptyset$, $N_{P_1}(w_0)^{-j} \cap N_{P_1}(u_p) = \emptyset$, $1 \leq j \leq l$ (by Operations 14 and 15),

$N_{P_1}(v_0) \cap N_{P_1}(w_0)^{+j} = \emptyset$ and $N_{P_1}(v_0) \cap N_{P_1}(w_0)^{-j} = \emptyset$, $1 \leq j \leq l$ (by Operation 16),

$N_{P_2}(w_0)^{+j} \cap N_{P_2}(v_0) = \emptyset$ and $N_{P_2}(w_0)^{-j} \cap N_{P_2}(v_p) = \emptyset$, $1 \leq j \leq l$ (by Operations 18 and 19),

$N_{P_2}(w_0) \cap \{v_{q-l+1}, ..., v_q\} = \emptyset$ (by Operation 20).

Moreover, if $u_i, u_j \in N_{P_1}(v_0)$, $i < j$, by Operations 9 and 11, $P(u_i, u_i^{+q}] \cap (N_{P_1}(u_0) \cup N_{P_1}(u_p)) = \emptyset$ or by Operations 10 and 11, $P(u_j^{-q}, u_j] \cap (N_{P_1}(u_0) \cup N_{P_1}(u_p)) = \emptyset$.

If $N_{P_1}(v_0) \cup N_{P_1}(v_q) = \emptyset$ and $v_0v_q \notin E(G)$, then $\bar{\sigma}_4 \leq d(u_0) + d(u_p) + d(v_0) + d(v_q) - |N(u_0) \cap N(u_p) \cap N(v_0) \cap N(v_q)| \leq |P_1| - 2 + |P_2| - 2 = |P_1| + |P_2| - 4$. Thus $|P_1| + |P_2| \geq \bar{\sigma}_4 + 4$. If $v_0v_q \in E(G)$, by the connectivity of G and the choice of P_2, a vertex of P_2 is adjacent to a vertex of P_1. As $G[V(P_2)]$ contains a Hamilton cycle, assume that $N_{P_1}(v_0) \neq \emptyset$. By Operation 12, $N_{P_1}(u_0)^{-j} \cap N_{P_1}(u_q) = \emptyset$, $1 \leq j \leq q$. Similarly, if $N_{P_1 \cup P_2}(w_0) = N_{P_1 \cup P_2}(w_l) = \emptyset$ and w_0 is non-adjacent to w_l, $|P_1| + |P_2| \geq \bar{\sigma}_4 + 4$. If $w_0w_l \in E(G)$, $G[V(P_3)]$ contains a Hamilton cycle. By the connectivity of G, assume $N_{P_1 \cup P_2}(w_0) \neq \emptyset$. If $N_{P_1}(w_0) = \emptyset$, by Lemma 2.2 and Lemma 2.3, $d(u_0) + d(u_p) + d(v_0) + d(w_0) - |N(u_0) \cap N(u_p) \cap N(v_0) \cap N(w_0)| \leq |P_3| - 1 + |P_1| + |P_2| - l = |P_1| + |P_2| - 1$. Thus $|P_1| + |P_2| \geq \bar{\sigma}_4 + 1$.

So assume that $N_{P_1}(w_0) \neq \emptyset$. If there exist two vertices $u_i \in N_{P_1}(v_0)$ and $u_j \in N_{P_1}(w_0)$, $i \neq j$, by Operation 16, $|j - i| > q$. Without loss of generality, we choose two such vertices u_i, u_j $(j > i)$, such that $\{u_{i+1}, u_{i+2}, ...u_{j-1}\} \cap (N_{P_1}(v_0) \cup N_{P_1}(w_0)) = \emptyset$. By Operation 17, $\{u_i\}^{+m} \cap N_{P_1}(u_p) = \emptyset$, $1 \leq m \leq l$. Take $P_1' = v_qP_2v_0u_iu_{i-1} \cdots u_0$ and

$P_2' = w_l P_3 w_0 u_j u_{j+1} \cdots u_p u_l u_{l+1} \cdots u_{j-1}$, where $u_l \in N_{P_1}(u_p) \cap P(u_i, u_j)$. By Lemma 2.1, $d(u_0)+d(u_p)+d(v_0)+d(w_0) \leq |P_1'|+2+\lambda_1$ and $d(u_0)+d(u_p)+d(v_0)+d(w_0) \leq |P_2'|+2+\lambda_2$. Then $\sum_{i=1}^4 d(u_0)+d(u_p)+d(v_0)+d(w_0) \leq |P_1'|+2+\lambda_1+|P_2'|+2+\lambda_1 = |P_1|+2-l+\lambda$, where $\lambda = |\cap_{i=1}^4 N_P(u_i)|$. Then $\sum_{i=1}^4 d(u_i) \leq |P_1|+2-l+\lambda+|P_2|-1+|P_3|-1 \leq |P_1|+|P_2|+\lambda$. So $|P_1|+|P_2| \geq \bar{\sigma}_4$. If $N_{P_2}(w_0) \neq \emptyset$, or $|N_{P_1}(v_0)| \geq 2$, or $|N_{P_1}(w_0)| \geq 2$, the result holds similarly as above. So $N_{P_1}(v_0) = N_{P_1}(w_0) = \{u_i\}$.

By the symmetry of v_0 and v_q, $N_{P_1}(v_q) = \{u_i\}$. If $v_0 v_q \notin E(G)$, $d(u_0)+d(u_p)+d(v_0)+d(v_q) \leq |P_1|+\lambda+|P_2|-2$. Then $|P_1|+|P_2| \geq \bar{\sigma}_4+2$. So assume that $G[V(P_2)]$ contains a Hamilton cycle. Then for any vertex v_i of $V(P_2)$, $N_{P_1}(v_i) \subseteq \{u_i\}$. Similarly for other end-vertices of P_i. Hence $G = \cup_{i=1}^k G_i$ such that $V(G_i) \cap V(G_j) = \{u_i\}$, for any $i, j \in \{1, 2, .., k\}$ $(k \geq 3)$. □

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