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# SOME RESULTS ON (p,g,sigma)-VALUATION OF GRAPHS

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# Some results on $(p, g, \Sigma)$-valuation of graphs

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**Abstract** : In this article we study proper labelings (or valuations) of the edges of a graph by integers, such that the sums of the values taken on the edges incident to each vertex ( the *weight* of this vertex) are all distinct. We look for the minimum of $\max(v)$ among the possible valuations $v$ for some particular classes of graphs.

**Keywords** : Graphs, paths, cycles, complete graphs, "vertex-distinguishing" valuations.

## I. INTRODUCTION.

We consider graphs $G = (V, E)$ without loops, multiple edges or isolated vertices, and consider valuation on the edges that allow to distinguish the vertices. There are several variations of this problem we describe below.

Consider a valuation $v$ on the edges of $G$ i.e. a function (otherwise a coloring) $v : E(G) \to \mathbf{N}^*$. We call $|v(E)|$ the size of the valuation. This coloring is (or not) proper.

It induces a valuation (or rather coloring) of the vertices of $G$ by

- either the sums $w(x) = \sum_{y \in N(x)} v(xy)$

- or the multisets $S(x) = \{v(xy), y \in N(x)\}$.

Then we add a last restraint

- either two adjacent vertices must have different colorings (local problem)

- or all the vertices must have different colorings (global problem).

This leads to eight possible notions and parameters :

$(p/np, l/g, \Sigma/\Omega)$ that is proper/no proper valuation of the edges in which the vertices are distinguished in a local/global manner with sums/sets and then correspondant parameters $\chi'_{\Omega/\Sigma}(G, p/np, l/g)$ : minimun of the maximum value in a $(p/np, l/g, \Sigma/\Omega)$-valuation. Several authors worked on some of these parameters, for instance see [1], [4], [3], [5].

In this article, we focus on $\chi'_\Sigma(G, p, g)$. Call admissible valuation on $E$ any proper valuation $v : E \to \mathbf{N}^*$ distinguishing vertices by sums, that is to say such that for any two vertices $x \neq y$, $\sum_{xz \in E} v(xz) \neq \sum_{yz \in E} v(yz)$. There is no such valuation for the graph $K_2$ so from now on, we assume that $|V(H)| \geq 3$ for every connected component $H$ of $G$.

An admissible valuation of size $|E| = m$ always exists : namely, let $E = \{e_1, e_2, \cdots, e_m\}$ and $v(e_i) = 2^{i-1}$. Since, for $x \neq y$, the set of edges incident to $x$ is distinct from the set of edges incident to $y$, the binary numbers $\sum_{x \in e_j} 2^{j-1}$ and $\sum_{y \in e_j} 2^{j-1}$ are distinct. Of course, this valuation is very bad in the sense that its maximum is by far too large, even if $m$ values are necessary. For instance, if $G = K_{1,n},\ n \geq 2$, any two edges are adjacent so every admissible valuation is of size $m = n$, but the values from 1 to $n$ are sufficient for distinguishing vertices by sums.

For $v$ an admissible valuation, and $x \in V$, we call *weight* of $x$ and note $w(x)$ the sum $w(x) = \sum_{x \in e_j} v(e_j)$.

The subset $\{\max(v(E)) \mid v \text{ an admissible valuation on } E\}$ of $\mathbf{N}$ being nonempty has a minimum, which we denote by $\chi'_w(G)$ for simplicity. For instance, $\chi'_w(K_{1,n}) = n$.

Recall that a proper vertex-distinguishing coloring (in short pvdc) of $E$ of size $q$, is a surjective application $\varphi : E \to \{1, \cdots, q\}$ with the following properties :

- for any two adjacent edges $e, e',\ \varphi(e) \neq \varphi(e')$
- for any two distinct vertices $x \neq y$ the multisets $\{\varphi(e) \mid x \in e\}$ and $\{\varphi(e) \mid y \in e\}$ are distinct.

Then we have the following :

**Theorem 1** *There is an admissible valuation on $E$ of given size, if and only if there is a pvdc of $E$ of the same size.*

PROOF. For a necessary condition, if $v : E \to \mathbf{N}^*$ is an admissible valuation of size $q$, then any bijection $g : v(E) \to \{1, \cdots, q\}$ induces a pvdc $\varphi = g \circ v$. Conversely, if $\varphi$ is a pvdc of size $q$, then $v = 2^{\varphi - 1}$ is an admissible valuation of the same size. ■

The minimum of colors used in a pvdc of $E$ is denoted by $\chi'_s(G)$. We immediately deduce the following :

**Corollary 1** *For any graph, we have $\chi'_s(G) \leq \chi'_w(G) \leq 2^{\chi'_s(G)-1}$.*

These bounds are tight. For instance, $\chi'_s(K_{1,n}) = \chi'_w(K_{1,n}) = n$ and $\chi'_w(G) = 4 = 2^{\chi'_s(G)-1}$ for the "extended-3-star" $G$ obtained by identifying the first extremities of three copies of $P_3$.

As an application of this corollary, if we find an admissible valuation $v$ on $E$ such that $\max(v) = \chi'_s(G)$ then we have $\chi'_w(G) = \chi'_s(G)$.

## II. SOME BOUNDS FOR $\chi'_w(G)$.

We give a lower bound for $\chi'_w(G)$ in the general case, and other bounds for regular graphs.

**Theorem 2** *If $G$ is a graph of order $n$, with maximum (respectively minimum) degree $\Delta$ (respectively $\delta$) then*

$$\chi'_w(G) \geq \left\lceil \frac{n-1}{\Delta} + \frac{\Delta - 1}{2} + \frac{\delta(\delta+1)}{2\Delta} \right\rceil .$$

PROOF. For any admissible valuation $v$ on $E(G)$, there are $n$ distinct weights on the vertices, so the minimum weight $w$ and the maximum weight $W$ satisfy the inequality $n - 1 \leq W - w$. On one hand we have in all cases $w \geq 1 + \cdots + \delta = \delta(\delta+1)/2$. On the other hand, if we have $\max(v) = \chi'_w(G)$, then $W \leq (\chi'_w(G) - \Delta + 1) + \cdots + \chi'_w(G) = \Delta(2\chi'_w(G) - \Delta + 1)/2$ thus $n - 1 \leq W - w \leq \Delta(\chi'_w(G) - (\Delta - 1)/2 - \delta(\delta+1)/(2\Delta))$ implying inequality of the theorem. ■

This bound is tight : for instance, we shall show that $\chi'_w(K_{p,p-1}) = p+1$ if $3 \leq p \leq 8$.

Let $G$ be a $d$-regular graph, $d \geq 2$, and $q$ any integer$\geq 1$. A valuation $v$ on $E$ is admissible if and only if the valuation $v + q$ is admissible, since all the weights are increased by $dq$. Therefore we have :

**Proposition 1** *If $G$ is regular and $v$ is an admissible valuation on $E$ with $\max(v) = \chi'_w(G)$, then $\min(v) = 1$.*

The following result is almost as obvious :

**Proposition 2** *If $G$ is a $d$-regular graph $\chi'_w(G\Box K_2) \leq 2\chi'_w(G) - d + 2$.*

PROOF. Recall that the cartesian product $G_1\Box G_2$ of two graphs is the graph $G$ such that $V(G) = V(G_1) \times V(G_2)$ and two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

Let $v$ be an admissible valuation on the edges of $G$ with maximum $\chi'_w(G)$. On one copy of $G$ put $v + 1$ so that the minimum is now 2. Since the difference between the maximum and the minimum weights is at most $d(\chi'_w(G) - d)$, by setting $v + (\chi'_w(G) - d + 2)$ on the edges of the second copy of $G$ we obtain distinct weights greater than those of the first copy. Now we give value 1 to the edges of the perfect matching corresponding to factor $K_2$ of the product and we are done. ■

We may slightly improve, for $d$-regular graphs, the lower bound $d + (n-1)/d$ given in the first theorem of this section by the following result, which is significant when $d$ divides $n - 1$ :

**Theorem 3** *Let $G$ be a $d$-regular graph of order $n$. Then we have :*

$$\chi'_w(G) \geq \left\lceil d + \frac{n-1}{d} + \frac{2\epsilon}{nd} \right\rceil$$

*with $\epsilon = 1$ if the number $n(d(d+1) + n - 1)/2$ is odd, and $\epsilon = 0$ otherwise.*

PROOF. Let $v$ be an admissible valuation on $E$ with $\max(v) = \chi'_w(G) = p$, and size $|v(E)| = q$, say $v(E) = \{v_1, \ldots, v_q\}$. For $1 \leq i \leq q$, let $k_i$ be the number of edges such that $v(e) = v_i$, so we have $\sum_{i=1}^{q} k_i = |E| = nd/2$. The $n$ weights $w(x)$, $x \in V$ are distinct numbers at least equal to $D = 1 + \cdots + d$. So the total sum of weights is at least $D + \cdots + (D + n - 1) = n(2D + n - 1)/2$. In this sum, the value $v_i$ appears $2k_i$ times, therefore we obtain, since this sum is even ($\epsilon$ being as in the statement of the Theorem) :

$$2\sum_{i=1}^{q} k_i v_i \geq \epsilon + n(2D + n - 1)/2.$$

Now, since $G$ is regular, $v' = p + 1 - v$ is another admissible valuation on $E$ and we have also :

$$2\sum_{i=1}^{q} k_i v'_i \geq \epsilon + n(2D + n - 1)/2.$$

Adding these two inequalities, we obtain : $2(p+1)nd/2 \geq 2\epsilon + n(2D + n - 1)$ which gives the result. ■

This bound is tight : for instance if $G$ is the well-known Petersen graph, one can easily find an admissible valuation $v$ on its edges with $\max(v) = 7$.

We give now an upper bound for the parameter $\chi'_w$ of two disjoint copies of a regular graph. We use the symbol $\cup$ to denote the disjoint union.

**Proposition 3** *Let $G$ be a $d-$regular graph. Then we have :*
$\chi'_w(G \cup G) \leq 2\chi'_w(G) - d + 1.$

PROOF. Note $G_1$ the first copy and put on $E(G_1)$ an admissible valuation$v_1$ with $\max(v_1) = \chi'_w(G)$. The maximum possible weight is $W_1 = d(2\chi'_w(G) - d + 1)/2$. Now on the edges of the second copy put $v_1 + \chi'_w(G) - d + 1$ giving as minimum weight $d(d+1)/2 + d(\chi'_w(G) - d + 1) = d(2\chi' - w(G) - d + 3)/2 > W_1$ ■

We shall see that this bound is tight, for instance with $G = K_n$, $n$ odd, or $G = C_{4k+3}$.

The four next sections are devoted to some families of connected graphs.

## III. RESULTS FOLLOWING CONSTRUCTIONS FOR $\chi'_s$.

The construction given in [2] for a proper vertex-distinguishing coloring of the edges of $K_n$ of size $\chi'_s(K_n)$ altogether gives an admissible valuation :

**Theorem 4** *We have :*

$$\chi'_w(K_n) = \chi'_s(K_n) = \begin{cases} n & if\ n\ is\ odd \\ n+1 & if\ n\ is\ even \end{cases}$$

PROOF. Recall the construction of [2]. For $k \geq 2$ arrange the vertices of $K_{2k}$ in the form of a regular $(2k-1)$-gon $x_1, \ldots, x_{2k-1}$ with one vertex $x_{2k}$ in the center. The radial edge $(x_{2k}x_i)$ together with the edges perpendicular to it is a perfect matching, to which we give the valuation $i$. At this step, all the vertices have the same weight.

In order to obtain a $K_{2k-1}$ delete vertex $x_1$. Since the valuation was proper, the weights of the other vertices decrease by distinct values, which gives the result for $n$ odd.

Now, for $k \geq 3$, delete moreover $x_2$. It is easy to check that the sums $(v(x_ix_1) + v(x_ix_2))_{3 \leq i \leq 2k}$ are all distinct. Therefore we obtain an admissible valuation of $K_{2k-2}$ and the result for $n$ even. ■

## IV. SOME RESULTS ON IRREGULAR BIPARTITE COMPLETE GRAPHS.

We already saw that for $n \geq 2$, $\chi'_s(K_{n,1}) = \chi'_w(K_{n,1}) = n$ with the set of values $\{1, \ldots, n\}$ on the edges. So we focus on the graphs $K_{n,p}$ with $n > p \geq 2$.

We shall denote by $x_i$ the vertices of one class (if $n \neq p$, the larger one) and by $x'_j$ those of the other one. Following the process which leads to $\chi'_s(K_{n,p}) = n + 1$ [2] , we may take a $K_{n+1,n+1}$ and put, for $1 \leq i \leq n+1$ and $0 \leq j \leq n$ on each edge $x_i x'_{j+i}$ (or $x_i x'_{j+i-n-1}$ if $i+j > n+1$) the value $v_{j+1}$ in such a way that the set $\{v_i \mid 1 \leq i \leq n+1\}$ equals $\{1, \ldots, n+1\}$, then erase one vertex $x_i$ of the first class, and $n+1-p$ vertices of the other class. Unfortunately, this does not give distinct weights in general. However we have :

**Theorem 5** *If $p$ is relatively prime to $n+1$, and $2 \leq p \leq n-3$, then $\chi'_w(K_{n,p}) = \chi'_s(K_{n,p}) = n+1$.*

PROOF. For any integer $k$, let $\overline{k}$ be the unique integer in the range $[1, n+1]$ such that $k - \overline{k}$ is divisible by $n+1$. Let $q = n+1-p$, so $q$ is relatively prime to $n+1$. Put $a = n/2$ if $n$ is even, $a = (n-1)/2$ if $n = 4k+3$ and $a = (n-3)/2$ if $n = 4k+1$. In every case $a$ is relatively prime to $n+1$.

With the above notations, let $v_i = \overline{1 + (i-1)a}$. Since $a$ is relatively prime to $n+1$, the sets $\{v_i \mid 1 \leq i \leq n+1\}$ and $\{j \mid 1 \leq j \leq n+1\}$ are equal, so the weights on the edges of the $K_{n+1,n+1}$ are all equal to $W = 1 + \cdots + (n+1) = (n+1)(n+2)/2$. Now we erase the vertex $x_1$ in the first class, and vertices $x'_i, p+1 \leq i \leq n+1$ in the other class.

Therefore the weights of the vertices $x'_i, 1 \leq i \leq p$ decrease respectively by the values $v_i$, all distinct and all no greater than $n+1$ and the remaining weights $w'_i$ are therefore all distinct. On the other hand, the weights of the $x_i$ decrease since $q \geq 4$ at least by $\overline{1-a} + 1 + (1+a) = n+4$ and the remaining weights $w_i$ are all distinct from the $w'_i$. For $1 \leq i \leq n+1$, let $s_i = (1 + (i-1)a) + (1 + ia) + \cdots + (1 + (i+q-2)a)$ and $\tilde{s}_i = v_i + \cdots + v_{\overline{i+q-1}}$, so for $1 \leq i < j \leq n+1$, $s_j - s_i - (\tilde{s}_j - \tilde{s}_i)$ is divisible by $n+1$, whereas $s_j - s_i = (j-i)qa$ is not, since $qa$ is relatively prime to $n+1$. Thus the $\tilde{s}_i$ are all distinct. Now the weights $w_i$ are $n$ distinct elements in the set $\{W - \tilde{s}_i \mid 1 \leq i \leq n+1\}$, so we obtain an admissible valuation on the edges of $K_{n,p}$. ■

With other choices of the values $v_i$, we obtain the following

**Theorem 6** *For any $n \geq 4$, $\chi'_w(K_{n,n-2}) = \chi'_s(K_{n,n-2}) = n+1$.*

PROOF. As above, let $W = 1+\cdots+(n+1)$ and $\{v_i \mid 1 \leq i \leq n+1\} = \{1, \ldots, n+1\}$. For any choice of the values $v_i$, by erasing vertices $x_{n+1}, x'_{n-1}, x'_n$ and $x'_{n+1}$, the remaining weights for the other $x'_i$ are all distinct and not smaller than $W - (n+1)$ ; those of the vertices $x_i, 1 \leq i \leq n-1$ are the elements of the set $\{W - (v_i = v_{i+1} + v_{i+2}) \mid 1 \leq i \leq n-1$ and that of the vertex $x_n$ is $W - (v_{n+1} + v_1 + v_2)$. In order to obtain an admissible valuation, it is sufficient that the $n$ sums $v_i + v_{i+1} + v_{i+2}, 1 \leq i \leq n-1, v_n + v_{n+1} + v_1$ are distinct and all greater than $n+1$. We give in any case a choice satisfying these properties, letting the checking to the reader.

- If $n = 3k-2$, for $1 \leq i \leq k, v_{3i-2} = i, v_{3i-1} = i+k$ and for $1 \leq i \leq k-1, v_{3i} = i+2k$.
- If $n = 3k-1$, for $1 \leq i \leq k, v_{3i-2} = i-1+2k, v_{3i-1} = i$, for $1 \leq i \leq k-1, v_{3i} = i+k$ and $v_{3k} = 3k$.

- If $n = 3k$, for $1 \leq i \leq k, v_{3i-2} = i, v_{3i-1} = i + k, v_{3i} = i + 2k$ and $v_{n+1} = n + 1$.

■

And with some slight modifications, we also obtain the following result :

**Theorem 7** *If $p$ satisfies $2 \leq p < n - (\sqrt{8n+25} - 5)/2$, then $\chi'_w(K_{n,p}) = \chi'_s(K_{n,p}) = n + 1$.*

PROOF. We only need to give a proof when $gcd(n+1, p) = d$ is at least 2 and $p \geq 3$. First begin with valuations $v_i = i$ on the edges of a $K_{n+1,n+1}$, and weight $W = (n+1)(n+2)/2$ for all its vertices. Then erase vertex $x_1$ of the first class and vertices $x_i, p+1 \leq i \leq n+1$ of the other one. The remaining weights of the second class are $W - i, 1 \leq i \leq p$, all distinct. Those of the first class are the elements of the two sets $\mathcal{W}_1 = \{w_i = i + \cdots + (i+p-1) \mid 2 \leq i \leq n-p+2\}$, and $\mathcal{W}_2 = \{w_j = j + \cdots + (n+1) + 1 + \cdots + (j+p-n-2) = j + \cdots + (j+p-1) - (j+p-n-2)(n+1) \mid n-p+3 \leq j \leq n+1\}$. The elements of each $\mathcal{W}_i$ are obviously distinct, but it may occur that some $w_i$ in $\mathcal{W}_1$ is equal to some $w_j$ in $\mathcal{W}_2$, which actually is the case. Now add 1 to the valuations $v(x_i x'_p)$ for $2 \leq i \leq p$ and substract $p$ to the valuation $v(x_{n+1} x'_p)$ (the result is 1). By this modification, the weights of the second class become $W - i, 1 \leq i \leq (p-1)$ and $W - (p+1)$. those of $\mathcal{W}_1$ remain unchanged except for the lesser one $w_2$ replaced by $w_2 - p$ (so these weights remain distinct) and each weight of $\mathcal{W}_2$ is increased by 1, and they remain distinct.

Let define $d'$ by :

- $d' = d$ if $d$ is odd or if $d = 2$ and $p$ divisible by 4.
- $d' = d/2$ otherwise.

Note that we have $d' \geq 2$ except for the case when $d = 2$ and $p$ not divisible by 4, where $d' = 1$. Since each sum $i + \cdots + (i+p-1) = p(2i+p-1)/2$ is divisible by $p$ if $p$ is odd, and by $p/2$ but not by $p$ if $p$ is even, each weight of $\mathcal{W}_1 \cup \mathcal{W}_2$, before modification is divisible by $d'$ when $d' \geq 2$ (respectively is odd when $d' = 1$) and after modification, this property is preserved for the weights of $\mathcal{W}_1$ but not for the weights of $\mathcal{W}_2$ therefore these modified weights are all distinct. Now, condition $p < n - (\sqrt{8n+25} - 5)/2$ insures that they are distinct from the remaining weights of the other class, since they are at most equal to $(n-p+2) + \cdots + (n+1) = p(2n-p+3)/2$ and the equation $p(2n-p+3) = 2(W - (p+1))$ (that is to say $p^2 - (2n+5)p + (n^2+3n) = 0$) in $p$ has two roots, namely $p_1 = n - (\sqrt{8n+25} - 5)/2$ and $p_2 = n + (\sqrt{8n+25} - 5)/2 > n$. ■

CONJECTURE. For $2 \leq p \leq n - 2$ we always have $\chi'_w(K_{n,p}) = n + 1$.

On the contrary, whereas for any $n \geq 3$, $\chi'_s(K_{n,n-1}) = n + 1$, we have

**Theorem 8** *If* $3 \leq n \leq 8$, $\chi'_w(K_{n,n-1}) = n+1$, *but for* $n \geq 9$, $\chi'_w(K_{n,n-1}) \geq n+2$.

PROOF. We can put the values of a valuation on the edges of a $K_{n,p}$ as entries of a $(n,p)-$ matrix $V$, namely $v_{(i,j)} = v(x_i x'_j)$. The valuation is admissible if and only if the entries in each row or column of $V$ are all distinct and the $n+p$ sums of entries of a row or column are all distinct. For $3 \leq n \leq 8$ the following matrices give an admissible valuation with maximum value $n+1$ for the edges of $K_{n,n-1}$ :

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 1 & 2 \\ 4 & 5 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 1 & 6 & 3 \\ 5 & 6 & 4 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 4 & 5 & 7 \\ 2 & 6 & 7 & 1 & 3 \\ 3 & 1 & 6 & 2 & 4 \\ 4 & 2 & 1 & 3 & 5 \\ 5 & 7 & 3 & 4 & 6 \\ 6 & 4 & 5 & 7 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 8 & 3 & 6 & 4 & 2 \\ 2 & 1 & 8 & 7 & 5 & 3 \\ 3 & 5 & 1 & 8 & 6 & 4 \\ 4 & 3 & 2 & 1 & 7 & 5 \\ 5 & 4 & 7 & 2 & 8 & 6 \\ 6 & 2 & 4 & 3 & 1 & 7 \\ 7 & 6 & 5 & 4 & 3 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 8 & 9 & 6 & 4 & 3 \\ 2 & 5 & 9 & 8 & 7 & 6 & 4 \\ 3 & 9 & 1 & 7 & 8 & 5 & 2 \\ 4 & 3 & 2 & 1 & 9 & 7 & 5 \\ 5 & 4 & 3 & 2 & 1 & 8 & 6 \\ 6 & 1 & 4 & 3 & 2 & 9 & 7 \\ 7 & 6 & 5 & 4 & 3 & 1 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 9 \end{pmatrix}$$

Now if there exists an admissible valuation with maximum value $n+1$ on the edges of a $K_{n,n-1}$ the sums of the rows of the associate $(n, n-1)$-matrix are $n$ distinct elements of the set $S = \{w, \ldots, w + 2(n-1)\}$ where $w = 1 + \cdots + (n-1)$ and those of the columns are $n-1$ distinct elements of the set $G = \{w+n, \ldots, w+2n\}$, the sum $\sigma$ of the $n$ weights taken in $S$ being equal to the sum $\Sigma$ of the $n-1$ weights taken in $G$. Note that we have $G \setminus S = \{w + (2n-1), w + 2n\}$ and that $S \cap G$ is of cardinality $n-1$. Let $k$ the number of elements of the set $G \setminus S$ occuring in the sum $\Sigma$. If $k$ were 0, the elements occuring in $\sigma$ would all be in the set $S \setminus G$ and we would have $\sigma < \Sigma$, a contradiction. So $k = 1$ or $k = 2$. For $k = 1$ we have $\Sigma \geq (n-1)w + (2n-1) + (n-2)(3n-3)/2 = (n-1)w + 3n^2/2 - 5n/2 + 2$ and $\sigma \leq nw + (2n-2) + (n-1)n/2 = (n-1)w + n^2 + n - 2$. Equality $\Sigma = \sigma$ implies $3n^2/2 - 5n/2 + 2 \leq n^2 + n - 2$ so $n \leq 5$.

For $k = 2$ we obtain $\Sigma \geq (n-1)w + 4n - 1 + (n-3)(3n-4)/2 = (n-1)w + 3n^2/2 - 5n/2 + 5$ and $\sigma \leq nw + 4n - 5 + (n-2)(n+1)/2 = (n-1)w + n^2 + 3n - 6$ and equality between the sums implies $3n^2/2 - 5n/2 + 5 \leq n^2 + 3n - 6$ so $n \leq 8$.

The following matrix actually gives an admissible valuation on the edges of $K_{9,8}$ with maximum value 11.

$$
\begin{pmatrix}
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
5 & 11 & 9 & 8 & 10 & 2 & 1 & 6 \\
6 & 10 & 7 & 9 & 1 & 8 & 11 & 4 \\
7 & 3 & 8 & 6 & 11 & 1 & 2 & 10 \\
8 & 9 & 10 & 1 & 2 & 7 & 4 & 3 \\
9 & 2 & 1 & 10 & 4 & 11 & 3 & 5 \\
10 & 4 & 11 & 2 & 3 & 6 & 5 & 9 \\
11 & 1 & 4 & 5 & 7 & 3 & 6 & 2 \\
3 & 8 & 5 & 11 & 9 & 10 & 7 & 1
\end{pmatrix}.
$$

■

CONJECTURE. For $n \geq 9$ we have $\chi'_w(K_{n,n-1}) = n + 2$.

## V. THE REGULAR BIPARTITE COMPLETE GRAPHS.

**Theorem 9** *For $n \geq 2, \chi'_w(K_{n,n}) = \chi'_s(K_{n,n}) = n + 2$.*

PROOF. Let $v$ any valuation on the edges of a $K_{n,n}$ whose values are in the set $E_n = \{1, \cdots n+2\}$. As above, we set the values $v_{i,j} = v(x_i x'_j)$ as entries of an $(n, n)-$matrix $V$. Then $v$ is an admissible valuation if and only if the $n$ rows $L_i$ and the $n$ columns $V_i$ are $2n$ subsets (necessarily distinct) of cardinality $n$ of $E$ with the following properties :

- For any $k \in E$ the sets $\{i \mid k \in L_i\}$ and $\{j \mid k \in C_j\}$ have the same cardinality.
- The $2n$ sums of the entries of each row and each column are distinct.

Since the graph is regular, this is equivalent to the fact that the $2n$ complementary subsets $L'_i = L_i^c$ and $C'_j = C_j^c$ are $2n$ subsets of $E$ of cardinality 2 satisfying the same properties.

Thus we can solve the problem in two steps : first give $2n$ subsets of cardinality 2 in $E$ having the required properties, then construct an $(n, n)$-matrix $V$ such that the sets $L'_i$ (respectively $C'_j$) are the sets of "missing numbers" in the rows (respectively the columns) of $V$. This is done in the following for $n \geq 5$ since the following matrices are easily seen as solutions for respectively $n = 2, 3$ and $4$ :

$$
\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}
\quad
\begin{pmatrix} 5 & 2 & 4 \\ 3 & 1 & 2 \\ 4 & 5 & 1 \end{pmatrix}
\quad
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 5 \\ 5 & 6 & 4 & 1 \\ 6 & 4 & 5 & 2 \end{pmatrix}.
$$

FIRST STEP. If $n$ is even, take as sets $L'_i$ the $n/2$ sets $\{1, i\}$ with $2 + n/2 \leq i \leq n + 1$ together with the $n/2$ sets $\{j, n+2\}$ with $2 \leq j \leq 1+n/2$. As sets $C'_j$ the $n/2$ sets $\{1, i\}$ with $2 \leq i \leq 1+n/2$ together with the $n/2$ subsets $\{j, n+2\}$ with $2+n/2 \leq j \leq n+1$.

If $n$ is odd, for the $L'_i$ take the sets $\{1, i\}$, $(n+3)/2 \leq i \leq n$ and the sets $\{j, n+2\}$, $2 \leq j \leq (n-1)/2$ together with the sets $\{(n+1)/2, (n+3)/2\}$ and $\{n+1, n+2\}$. For the $C'_j$ take the sets $\{1,i\}, 2 \leq i \leq (n+1)/2$ and the sets $\{j, n+2\}, (n+3)/2 \leq j \leq n$ together with the set $\{(n+3)/2, n+1\}$.

In every case, the required properties are easy to check.

<u>SECOND STEP.</u> First define, for $k$ elements $(a_i)_{1\leq i\leq k}$, a matrix $C(a_1,\dots,a_k)$ by $\forall (i,j) \in \{1,\dots,k\}^2, c_{(i,j)} = a_{\overline{j-i+1}}$ where $\overline{s}$ is the unique integer in $\{1,\dots,k\}$ such that $s-\overline{s}$ is divisible by $k$.

We divide our construction into three cases.

FIRST CASE : $n$ ODD. Let $n = 2k+1$. Put $A_1 = A_2 = C(1,\dots,k+1)$ and $B_1 = B_2 = C(k+2,\dots,2k+2)$. In $A_1$ replace, for $1 \leq i \leq k+1$, $a_{(i,i)}$ (whose value is 1) by $n+2$, call $\tilde{A}_1$ this new matrix. In $B_2$ interchange the rows 1 and $k$, we obtain a new matrix $B'_2$ ; in this matrix, replace $b'_{(k,k+1)}$ by 1 and $b'_{(k+1,k+1)}$ by $n+2$, name $\tilde{B}_2$ the resultant matrix. Build with these matrices a $(2k+2,2k+2)$-matrix $V' = \begin{pmatrix} \tilde{A}_1 & B_1 \\ \tilde{B}_2 & A_2 \end{pmatrix}$. Now erasing row $k+1$ and column $2k+1$ of this matrix gives as result a matrix $V$ associated to an admissible valuation for the edges of a $K_{n,n}$.

For instance, if $n = 7$, the result is the following matrix

$$\begin{pmatrix} 9 & 2 & 3 & 4 & 5 & 6 & 8 \\ 4 & 9 & 2 & 3 & 8 & 5 & 7 \\ 3 & 4 & 9 & 2 & 7 & 8 & 6 \\ 7 & 8 & 5 & 6 & 1 & 2 & 4 \\ 8 & 5 & 6 & 7 & 4 & 1 & 3 \\ 5 & 6 & 7 & 1 & 3 & 4 & 2 \\ 6 & 7 & 8 & 9 & 2 & 3 & 1 \end{pmatrix}.$$

SECOND CASE : $n = 4k+2$. Put $A = C(1,\dots,2k+1), B = C(2k+2,\dots,4k+2)$. For $2 \leq i \leq 2k+1$, exchange $a_{(i,2k+3-i)}$ and $b_{(i,2k+3-i)}$ in order to obtain two new matrices $A'$ and $B'$. Make two copies $A'_1, A'_2$ of $A'$ and two copies $B'_1, B'_2$ of $B'$. In $A'_1$ replace, for $1 \leq i \leq 2k+1$, $a'_{(i,i)}$ (whose value is 1) by $n+2$, we obtain a matrix $\tilde{A}$. In $B'_1$, replace for $1 \leq i \leq k$ and $k+2 \leq i \leq 2k+1$, $b'_{(i,2k+2-i)}$ by $n+1$, we obtain the matrix $\tilde{B}_1$. In $B'_2$ replace $b'_{(1,1)}$ and, for $2 \leq i \leq 2k+1$, $b'_{(i,2k+3-i)}$ by $n+1$, we obtain a matrix $\tilde{B}_2$. Now the matrix $V = \begin{pmatrix} \tilde{A} & \tilde{B}_1 \\ \tilde{B}_2 & A'_2 \end{pmatrix}$ is associated to an admissible valuation on the edges of a $K_{n,n}$.

For instance, for $n = 10$ the result is the following matrix

$$\begin{pmatrix} 12 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 11 \\ 5 & 12 & 2 & 3 & 9 & 10 & 6 & 7 & 11 & 4 \\ 4 & 5 & 12 & 7 & 3 & 9 & 10 & 6 & 2 & 8 \\ 3 & 4 & 10 & 12 & 2 & 8 & 11 & 5 & 6 & 7 \\ 2 & 8 & 4 & 5 & 12 & 11 & 3 & 9 & 10 & 6 \\ 11 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 \\ 10 & 6 & 7 & 8 & 11 & 5 & 1 & 2 & 3 & 9 \\ 9 & 10 & 6 & 11 & 8 & 4 & 5 & 1 & 7 & 3 \\ 8 & 9 & 11 & 6 & 7 & 3 & 4 & 10 & 1 & 2 \\ 7 & 11 & 9 & 10 & 6 & 2 & 8 & 4 & 5 & 1 \end{pmatrix}.$$

Third case : $n = 4k$. Put $B = C(2k+1, \dots, 4k)$. Construct matrix $A$ by interchanging in $C(1, \dots, 2k)$, for $i = k$ and $i = 2k$ the entries $c_{(i,1)}$ and $c_{(i,k+1)}$. Exchange between $A$ and $B$ for $2 \leq i \leq k$, $a_{(i,2k+2-i)}$ with $b_{(i,2k+2-i)}$, for $k+1 \leq i \leq 2k-1$, $a_{(i,2k+1-i)}$ with $b_{(i,2k+1-i)}$, and at last $a_{(2k,2k-1)}$ with $b_{(2k,2k-1)}$, we obtain two matrices $A'$ and $B'$ of which we make two copies $A'_1, A'_2$ and $B'_1, B'_2$. In $A'_1$, replace for $1 \leq i \leq 2k, a'_{(i,i)}$ (whose value is 1) by $n+2$ name this matrix $\tilde{A}$. In $B'_1$ replace respectively, for $1 \leq i \leq k$, $b'_{(i,2k+1-i)}$ and for $k+1 \leq i \leq 2k-1$, $b'_{(i,2k-i)}$ by $n+1$, we obtain $\tilde{B}_1$. In $B'_2$ replace respectively for $2 \leq i \leq k$, $b'_{(i,2k+2-i)}$, for $k+1 \leq i \leq 2k-1$, $b'_{(i,2k+1-i)}$ and $b'_{(2k,2k-1)}$ by $n+1$ in order to obtain $\tilde{B}_2$. Now matrix $V = \begin{pmatrix} \tilde{A} & \tilde{B}_1 \\ \tilde{B}_2 & A'_2 \end{pmatrix}$ gives a solution for $K_{n,n}$.

For instance, if $n = 8$ we obtain :

$$V = \begin{pmatrix} 10 & 2 & 3 & 4 & 5 & 6 & 7 & 9 \\ 2 & 10 & 4 & 7 & 8 & 5 & 9 & 3 \\ 3 & 8 & 10 & 2 & 9 & 4 & 5 & 6 \\ 4 & 3 & 8 & 10 & 6 & 7 & 2 & 5 \\ 9 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ 8 & 5 & 6 & 9 & 2 & 1 & 4 & 7 \\ 7 & 9 & 5 & 6 & 3 & 8 & 1 & 2 \\ 6 & 7 & 9 & 5 & 4 & 3 & 8 & 1 \end{pmatrix}.$$

■

Remark

Applying Proposition 3, we get: $n+4 \leq \chi'_w(K_{n,n} \cup K_{n,n}) \leq n+5$.

## VI. RESULTS FOR PATHS AND CYCLES.

It is not always a valuation of minimum size $\chi'_s(G)$ which gives the minimum $\chi'_w(G)$. For instance, an admissible valuation of $G = C_{36}$ of size $\chi'_s(G) = 9$ induces as weights all the combinations $v_i + v_j, i \neq j$, and these sums must be all distinct, implying $\max(v) > 20$, whereas we show in this section that $\chi'_w(G) = 20$.

Let $G$ be a graph whose all connected components are paths, say $G = P_{n_1} \cup \dots \cup P_{n_k}$. If we close each path by an extra edge, we obtain a graph $\hat{G} = C_{n_1} \cup \dots \cup C_{n_k}$. Now, let $v$ be an admissible valuation on $E(G)$, put $\tilde{v} = v$ on the common edges $E(G) \cap E(\hat{G})$ and $\tilde{v} = 0$ on the extra edges. Therefore $\hat{v} = \tilde{v} + 1$ is an admissible valuation on $E(\hat{G})$. Thus we obtain :

**Lemma 1** *If $G = P_{n_1} \cup \dots \cup P_{n_k}$ and $\hat{G} = C_{n_1} \cup \dots \cup C_{n_k}$ then $\chi'_w(\hat{G}) \leq \chi'_w(G) + 1$* ■

Now, since $\hat{G}$ is regular, theorem 3 gives a lower bound $L$ for $\chi'_w(\hat{G})$ and if we find an admissible valuation $v$ on $E(G)$ with maximum $L-1$, we obtain $\chi'_w(G) = L-1$ and $\chi'_w(\hat{G}) = L$ by the valuation $\hat{v}$.

In the case when $G$ is connected, i.e. $k = 1$, theorem 3 gives (note that for $n = 4k+1$ we have $\epsilon = 1$) :

- $\chi'_w(C_n) \geq (n+4)/2$, if $n$ is even
- $\chi'_w(C_n) \geq (n+5)/2 = 2k+3$ if $n = 4k+1$
- $\chi'_w(C_n) \geq (n+3)/2 = 2k+3$ if $n = 4k+3$

Therefore, by the previous lemma, we obtain :

- $\chi'_w(P_n) \geq (n+2)/2$ if $n$ is even
- $\chi'_w(P_n) \geq (n+3)/2$ if $n = 4k+1$
- $\chi'_w(P_n) \geq (n+1)/2$ if $n = 4k+3,\ k \geq 2$

But in the special case $n = 7$, we have $\chi'_w(P_7) \geq 5$. In fact for an admissible valuation $v$ with $\max(v) = \chi'_w(P_7) = p$, the seven weights are distinct numbers, all at least 1. If the value $v_i$ of $v$ is attributed to $k_i$ edges, we have $2\sum k_i v_i \geq n(n+1)/2 = 28$ and $\sum k_i = 6$. With the above notation, since $\tilde{v}_0 = 0$, we have also $2\sum k_i \tilde{v}_i \geq 28$, now $\sum k_i = 7$. But $v' = p+1-\tilde{v}$ is an admissible valuation of $C_7$, thus $2\sum k_i v'_i \geq n(n+5)/2 = 42$. Adding these two inequalities gives $2 \times 7 \times (p+1) \geq 70$ thus $p \geq 4$. We can have equality only if the second inequality is an equality, which implies that the seven weights (on the vertices of $C_7$) are in fact the numbers from 1 to 7. But the only possible decompositions of six among them for $p = 4$ are : $1 = 0+1, 2 = 0+2, 3 = 1+2, 4 = 1+3, 6 = 2+4, 7 = 3+4$ whereas for 5 there are two possible decompositons. If we choose $5 = 1+4$ (respectively $5 = 2+3$), then the value 2 (resp. 1) would appear three times, a contradiction since there must be an even occurence of each value in the set of weights. ■

**Theorem 10** *Previous inequalities are equalities.*

PROOF. As explained above, it suffices to give in each case an admissible valuation $v$ on $E(P_n)$ with maximum equal to the previous lower bound given, or explicit this valuation by the list $(v)$ of its values taken on the edges in the natural order.

Since the line graph of a $P_n$ is a $P_{n-1}$ we may obtain this result by constructing a connected graph $\Gamma$ with $|E(\Gamma)| = n-2$ and $V(\Gamma) \subseteq \{1, \ldots, \chi'_w(\Gamma)\}$, two vertices $i_1, i_2$ being of odd degrees, the others of even degrees. Then $\Gamma$ has an eulerian path with origin $i_1$ and extremity $i_2$ inducing the required line graph.

- If $n = 4k,\ k \geq 1,\ E(\Gamma) = \{(1,i)_{3 \leq i \leq 2k+1}, (j, 2k+2)_{2 \leq j \leq 2k}\}$
- If $n = 4k+1,\ k \geq 1,\ E(\Gamma) = \{(1,i)_{3 \leq i \leq 2k+1}, (j, 2k+2)_{2 \leq j \leq 2k+1}\}$
- If $n = 4k+2,\ k \geq 1,\ E(\Gamma) = \{(1,i)_{2 \leq i \leq 2k+2}, (j, 2k+2)_{3 \leq j \leq 2k+1}\}$
- If $n = 4k+3,\ k \geq 2, E(\Gamma) = \{(1,i)_{2 \leq i \leq k+1}, (1,i)_{k+3 \leq i \leq 2k+1}, (2, k+1), (k+1, k+2), (j, 2k+2)_{2 \leq j \leq 2k+1}\}$

- If $n = 3$, (v)=(1,2)
- If $n = 7$, (v)=(1,2,3,1,5,2)

Note that in every case, the extremities of any eulerian path of $\Gamma$ are the vertices 2 and 1. Note also that in the case $n = 3$ where we don't use the result of the lemma, the list $(1, 2)$ and its symmetric $(2, 1)$ are the only possibilities of a proper valuation on the edges. We exhibit now possible sequences $(v)$ corresponding to eulerian paths of the given graphs $\Gamma$ :

- For $n = 4k$, $k \geq 1$, $(v) = (2, (2k + 1, 2i - 1, 1, 2i)_{2 \leq i \leq k}, 2k + 1, 1)$
- For $n = 4k + 1$, $k \geq 1$, $(v) = (2, (2k + 2, 2i - 1, 1, 2i)_{2 \leq i \leq k}, 2k + 2, 2k + 1, 1)$
- For $n = 4k + 2$, $k \geq 1$, $(v) = (2, 1, 3, (2k + 2, 2i, 1, 2i + 1)_{2 \leq i \leq k}, 2k + 2, 1)$.
- For $n = 4k + 3$, $k$ even,
  $(v) = (1, 2i, 2k + 2, 2i + 1)_{1 \leq i \leq k/2}, (1, 2j + 1, 2k + 2, 2j)_{k \geq j \geq k/2+1}, k + 1, 2)$.
- For $n = 4k + 3$, $k$ odd $\geq 3$,
  $(v) = (1, 2i, 2k + 2, 2i + 1)_{1 \leq i \leq k-1/2}, (1, 2j, 2k + 2, 2j + 1)_{k+3/2 \leq j \leq k}, (1, k + 1, 2k + 2, k + 2), k + 1, 2.$

■

**Corollary 2**

- $\chi'_w(C_n) = (n + 4)/2$, *if $n$ is even*
- $\chi'_w(C_n) = (n + 5)/2 = 2k + 3$ *if $n = 4k + 1$*
- $\chi'_w(C_n) = (n + 3)/2 = 2k + 3$ *if $n = 4k + 3$*

PROOF. According to the remark following the lemma, we use for $n \neq 7$ the valuation $\overline{v}$ on the edges of $C_n$. In the special case $n = 7$ we give a valuation by its list : $(v) = (5, 1, 2, 5, 3, 1, 4)$. ■

The last sections are devoted to some families of non connected graphs, without connected component isomorphic to $K_1$ or $K_2$. If $G_1, \ldots, G_k$ are connected graphs, we call $G_1 \cup \cdots \cup G_k$ a graph having the $G_i$ as connected components.

## VII. TWO COPIES Of THE SAME COMPLETE GRAPH.

**Theorem 11** *We have : $\chi'_w(K_n \cup K_n) = n + 2$.*

PROOF. According to theorem 3, $n+2$ is a lower bound. It remains to give an admissible valuation with maximum $n+2$, but the construction depends on the parity of $n$.

• First case : $n$ odd, $n = 2k-1$.

For $n = 3$, the two lists $(v) = (1, 2, 3)$, $v' = (3, 4, 5)$ give an admissible valuation with maximum $n+2$. We assume thereafter that we have $k \geq 3$.

For the first copy we use valuation given in section III. In the second copy denote $x'_i$ the vertex called $x_i$ in the first copy. Instead of the value 1 on the edges $x_j x_{n+2-j}$, $2 \leq j \leq k$ we put $2k$ on the edges $x'_j x'_{n+2-j}$, and on edge $x'_2 x'_{2k}$ put $2k+1$ instead of the value 2 on $x_2 x_{2k}$. Now the minimum weight in the first copy is $w(x_{2k-2}) = k(2k-1)-(2k-1)$ and the maximum $w(x_{2k}) = k(2k-1)-1$. Thus we have, if $j \neq 2$, $w(x'_j) = w(x_j)+2k-1$ and $w(x'_2) = w(x_2)+4k-2$. Since in the first copy the minimum and maximum weights are respectively $w(x_{2k-2}) = S-(2k-1)$ and $w(x_{2k}) = S-1$ where $S = k(2k-1)$, whereas $w(x_2) = S-(k+1)$, we obtain $w(x'_{2k-2}) = S > w(x_{2k})$ and $w(x'_2) = S+3k-3 > w(x'_{2k}) = S+2k-2$.

• Second case : $n$ even, $n = 2k-2$, $k \geq 3$.

For the first copy we use valuation given in section III. The weights of the vertices are, if $3 \leq j \leq 2k-2$, $v(x_j) = S-(j+k+1)$ where $S = k(2k-1)$ as above, that is to say the successive integers from $S-(3k-1)$ to $S-(k+4)$, then $w(x_{2k-1}) = S-(k+1)$ and $w(x_{2k}) = S-3$.

Now in the second copy, instead of value $k+1$ on the edges $(x_{k+1-i}, x_{k+1+i})_{1 \leq i \leq k-2}$ and $(x_{k+1}, x_{2k})$ put value 1 on the corresponding edges. Moreover, if $k \geq 4$ instead of 1 on the edges $(x_i, x_{2k+1-i})_{3 \leq i \leq k-1}$ put value $k+1$ on the corresponding edges of the second copy. Now the weights of the vertices of this copy are, for $j \in \{k, k+1, 2k-1, 2k\}$ , $w(x'_j) = w(x_j)+k-1$ which gives the integers $S-(k+3), S-(k+2), S-2, S+k-4$, and complete an admissible valuation of maximum $n+2$ for $k = 3$. Moreover, for $k \geq 4$ and $3 \leq j \leq k-1$ or $k+2 \leq j \leq 2k-2$, we obtain $w(x'_j) = w(x_j)+2k-1$ which gives the integers from $S-k \leq m \leq S-4$ and $S-1 \leq m \leq S+k-5$ and this valuation is therefore an admissible one with maximum $n+2$. ∎

## VIII. TWO COPIES OF THE SAME PATH OR CYCLE.

**Lemma 2** *We have the inequalities :*

$$n+2 \leq \chi'_w(C_n \cup C_n) \leq \chi'_w(P_n \cup P_n)+1 \ .$$

PROOF. The lower bound comes from Theorem 3. As explained in the proof of lemma 2, if we put on the edges of $P_n \cup P_n$ an admissible valuation $v$, close each path by an edge labelled 0 and add 1 to $v$, we obtain an admissible valuation for $C_n \cup C_n$, from which results the second inequality ∎

**Theorem 12** *We have :* $\chi'_w(P_3 \cup P_3) = 5$ *and for* $n \geq 4$ , $\chi'_w(P_n \cup P_n) = n+1$.

PROOF. According to the previous Lemma, $\chi'_w(P_3 \cup P_3) \geq 4$. But an admissible valuation with maximum 4 would be a bijection from $E$ unto $\{1, 2, 3, 4\}$ and the four numbers would be weights of the four end-vertices, which excludes the couples of values $(1,2)$ and $(1,3)$ for the edges of any copy. Now the remaining couple $(1,4)$ is also excluded since $1+4=2+3$ and we must have $\chi'_w(P_3 \cup P_3) \geq 5$. Since the valuation taking the following couple of values on the two copies : $(1,3),(2,5)$ is admissible, we have equality. Note that this valuation is the first example where there is a gap in the set $v(E)$.

For the small values of $n$, we give an admissible valuation of maximum $n+1$ by two lists of values (one for each copy) :

- for $n=4$ : $(1,3,5)$ and $(2,4,3)$
- for $n=5$ : $(1,3,6,5)$ and $(3,4,6,2)$
- for $n=6$ : $(1,3,5,1,2)$ and $(5,7,6,3,7)$
- for $n=7$ : $(1,5,7,8,5,4)$ and $(3,2,6,8,3,7)$
- for $n=8$ : $(2,5,3,9,4,5,1)$ and $(3,8,7,9,8,6,4)$.

For $n \geq 8$ we use the same method as in section VI, by constructing two connected graphs $\Gamma, \Gamma'$ of order $n-1$ with $V(\Gamma) \cup V(\Gamma') \subseteq \{m \in \mathbf{N} \mid 1 \leq m \leq n+1\}$ such that only two vertices $i_1, i_2$ of $\Gamma$ (respectively $i'_1, i'_2$ of $\Gamma'$) are of odd degrees, and the $2n$ values $i_1, i_2, i'_1, i'_2, (i+j)_{(i,j) \in E(\Gamma) \cup E(\Gamma')}$ are all distinct, then using an eulerian path of $\Gamma$ (resp. of $\Gamma'$) as line graph of the first (resp. the second) copy of $P_n$. This is done thereafter.

• First case : $n=4k,\ k \geq 2$. Take $E(\Gamma) = \{(1,i)_{3 \leq i \leq 2k+1},\ (i,k+1)_{3 \leq i \leq 2k+1}\}$, the values $i_1, i_2$ being 1 and $4k+1$, and $E(\Gamma') = \{(1,2),\ (1,i)_{2k+2 \leq i \leq 4k-2},\ (3,4k-1),\ (3,4k)\ (i,4k+1)_{2k+2 \leq i \leq 4k-1}\}$, the values $i'_1, i'_2$ being 2 and $4k$.

• Second case : $n=4k+1,\ k \geq 2$. Take $E(\Gamma) = \{(1,i)_{3 \leq i \leq 2k+1},\ (i,4k+2)_{2 \leq i \leq 2k+1}\}$, the values $i_1, i_2$ being 1 and 2, and $E(\Gamma') = \{(3,i)_{2k+1 \leq i \leq 4k-1},\ (i,4k+2)_{2k+3 \leq i \leq 4k-1},\ (2k+1,2k+2),\ (2k+3,4k+1),\ (4k+1,4k+2)\}$, the two values $i'_1, i'_2$ being 3 and $2k+3$.

• Third case : $n=4k+2,\ k \geq 2$. Take $E(\Gamma) = \{(1,i)_{2 \leq i \leq 2k},\ (2,2k+2),\ (i,2k+3)_{2 \leq i \leq 2k},\ (2k+2,2k+3)\}$, the two values $i_1, i_2$ being 1 and 2, and $E(\Gamma') = \{(2k+1,2k+3),\ (2k+1,i)_{2k+5 \leq i \leq 4k+3},\ (2k+2,4k+3),\ (2k+4,4k+2),\ (i,4k+3)_{2k+4 \leq i \leq 4k+1}\}$, the two values $i'_1, i'_2$ being $2k+2$ and $2k+3$.

• Last case : $n=4k+3,\ k \geq 2$. Take $E(\Gamma) = \{(1,i)_{3 \leq i \leq 2k+3},\ (i,4k+3)_{4 \leq i \leq 2k+3}\}$, the two values $i_1, i_2$ being 1 and 3, and $E(\Gamma') = \{(2,2k+3),\ (3,i)_{2k+4 \leq i \leq 4k+3},\ (i,4k+4)_{2k+3 \leq i \leq 2k+5 \cup 2k+7 \leq i \leq 4k+3}\}$, the two values $i'_1, i'_2$ being 2 and $2k+6$. ■

**Corollary 3** *For $n \geq 3$ we have $\chi'_w(C_n \cup C_n) = n+2$.*

PROOF. The value $n+2$ being the lower bound of the lemma, it suffices to give an admissible valuation with maximum $n+2$. Since $C_3 = K_3$ the result for $n=3$ is given in the previous section. For $n \geq 4$ we use the valuation $v$ given in the theorem on $E(P_n \cup P_n)$, we close each copy of $P_n$ by an edge labelled 0 and add 1 to this pseudo-valuation on $E(C_n \cup C_n)$, so we obtain an admissible valuation with maximum $n+2$. ■

## IX. SOME EXAMPLES OF UNION OF TWO PATHS OR CYCLES OF DISTINCT ORDERS.

When $\chi'_w(P_n \cup P_{n+1}) = n+1$ (respectively $\chi'_w(C_n \cup C_{n+1}) = n+2$) the function $w$ is a bijection between $V$ and the set $\{m \in \mathbf{N} \mid 1 \leq m \leq 2n+1\}$ (resp. $\{m \in \mathbf{N} \mid 3 \leq m \leq 2n+3\}$. Since the valuation of each edge occurs in exactly two weights, the sum of the weights must be even, so we must have $(2n+1)(n+1)$ (resp. $(2n+1)(n+3)$) even, which implies $n$ odd. This implies the following :

**Lemma 3** *If $n$ is even $\chi'_w(C_n \cup C_{n+1}) \geq \chi'_w(P_n \cup P_{n+1}) + 1 \geq n+3$* ■

Now we obtain :

**Theorem 13** *For $n \geq 3$ : if $n$ is odd, then $\chi'_w(P_n \cup P_{n+1}) = \chi'_w(C_n \cup C_{n+1}) - 1 = n+1$ and if $n$ is even $\chi'_w(P_n \cup P_{n+1}) = \chi'_w(C_n \cup C_{n+1}) - 1 = n+2$.*

According to the remark following lemma 1, it is sufficient to give the proof in the case of paths.

PROOF. We use two techniques, illustrating each of the following cases.

FIRST CASE : $n$ ODD.

For the small values, we give one list of values of $v$ for each component :

- For $n = 3$ : $(1,4)$, $(2,4,3)$
- For $n = 5$ : $(1,5,6,4)$, $(2,3,4,5,3)$.

In the general case, we use the techniques described in section VI, but here we need two eulerian graphs $\Gamma$ and $\Gamma'$, one for each component. This case divides into two subcases :

• First subcase : $n = 4k+3$ : $E(\Gamma) = \{(1,i)_{3 \leq i \leq 2k+3},\ (i, 4k+4)_{4 \leq i \leq 2k+3}\}$, $E(\Gamma') = \{(i, 4k+4)_{2k+4 \leq i \leq 4k+3}, (2, 2k+4), (4, 2k+3), (4, 2k+4), (2k+3, 2k+4),$ and if $k \geq 2,\ (3, i)_{2k+6 \leq i \leq 4k+3}\}$.

Thus possible lists for $k \geq 2$ are : $\Big((1, i+1, 4k+4, i)_{2k+2 \geq i \geq 4}, 1, 3\Big)$ and $\Big(2, 2k+4, 4, 2k+3, 2k+4, (4k+4, i, 3, i+1)_{2k+6 \leq i \leq 4k+2}, 4k+4, 2k+5\Big)$.

• Second subcase : $n = 4k+1$ : $E(\Gamma) = \{(1,i)_{3 \leq i \leq 2k+1},\ (i, 4k+2)_{2 \leq i \leq 2k+1}\}$, $E(\Gamma') = \{(1,i)_{2k+4 \leq i \leq 4k+1}, (i, 4k+2)_{2k+2 \leq i \leq 4k+1}, (3, 2k+1), (2k+1, 2k+2)\}$.

Thus possible lists are : $\Big((1, i, 4k+2, i+1)_{3\geq i\geq 2k}, 1, 2k+1, 4k+2, 2\Big)$ and $\Big(3, 2k+1, 2k+2, 4k+2, (i, 1, i+1, 4k+2)_{2k+4\leq i\leq 4k}, 2k+3\Big)$.

<u>Second case : $n$ even.</u>

Let $n = 2k$. We use another method, by cutting a labelled path of order $4k+1$ into two pieces of required orders, and correcting some values of the labelling in order to obtain an admissible valuation. This method divides into two subcases :

• First subcase : $k$ odd, $k = 2k' + 1$. Recall that we have the following list $(v) = (2, (2k+2, 2i-1, 1, 2i)_{2\leq i\leq k}, 2k+2, 2k+1, 1)$ for $P_{4k+1}$.

We erase the edge following the median vertex, which was labelled $2k' + 3 = k + 2$ (corresponding to $i = k'+2$). On the last edge of the sublist corresponding to $i = k'+1$ replace $k+1$ by $k+2$ and the last edge of the path, is labelled $k+2$ instead of 1. It is easy to verify that the set of new weights is the same as this of lost weights.

• Second subcase : $k$ even, $k = 2k'$. We still consider the given valuation of $P_{4k+1}$. In the sublist corresponding to $i = k'+1$, we erase the last edge, labelled $2k'+2 = k+2$, in the sublist corresponding to $i = k'$ we replace $k$ by $k+2$ and finally the last edge of the path is labelled $k+1$ instead of 1. Here also, the set of new weights is the same as this of lost weights.

<u>Remarks</u> : By using the last technique, erasing one or more edges labelled 1 and correcting some values, we can obtain :

$*\chi'_w(P_{4k-4i+4} \cup P_{4i-4}) = 2k+1$ for $i = 2, 3, \cdots$, beginning with the valuation of $P_{4k}$.

For example, with $k = 4$ and $i = 2$, by erasing the third of the edges labelled 1, we get as valuations :

$(2, 9, 3, 1, 4, 9, 5, 2, 7, 9, 6)$ and $(8, 9, 1)$.

$*$ By a similar method, we obtain :

$\chi'_w(P_{4k-4i+5} \cup P_{4i-4}) = 2k+2$ for $i = 2, 3, \cdots$, beginning with the valuation of $P_{4k+1}$.

$*$ By erasing more edges, we obtain:

$\chi'_w(P_{4r_1} \cup P_{4r_2} \cdots \cup P_{4r_s} \cup P_{4q+1}) = 2k+2$ where $k = r_1 + r_2 + \cdots + r_s + q = 2k+2$.

For example, with $r_1 = 1, r_2 = 2, q = 1$, we get as valuations :

$(2, 10, 3), (4, 10, 5, 2, 7, 10, 6)$ and $(8, 10, 9, 1)$.

$*$ On a cycle $C_n$, put the "complementary" valuation $v' = \chi'_w(C_n) + 1 - v$, erase all the edges labelled 1, and then decrease by one all the valuations of the remaining edges. We obtain an admissible valuation $v'$ for the remaining union $G'$ of paths. Actually its maximum gives the value of $\chi'_w(G')$, since reconnecting the paths of $G'$ by edges labelled 0 in order to recover $C_n$ and adding 1 to this pseudo-valuation, one can prove the inequality $\chi'_w(C_n) \leq \chi'_w(G') + 1$.

For instance, with $n = 17$, we obtain $\chi'_w(P_4 \cup P_4 \cup P_4 \cup P_5) = 10 = \chi'_w(C_{17}) - 1$ with valuations $(5, 9, 4), (6, 9, 5, 7), (2, 9, 3)$ and $(8, 10, 9, 1)$.

<u>Problems</u>

- Characterize the graphs $G$ such that $\chi'_w(G) = \Delta(G)$ (Note that in this case, there is only one vertex of maximum degree).

- Is there a connected graph $G$ for which there is a gap in the sequence of integer values taken by any admissible valuation giving the value of $\chi'_w(G)$.

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